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Componentwise error estimates for approximate solutions to systems of equations with the aid of Dahlquist constants

Hisayoshi SHINTANI

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1. Introduction

For f: $D \subset R^n \to R^n$ consider the equation

$$x = f(x), \quad x \in D, \tag{1.1}$$

where D is a path-connected set. Let the l.u.b. Lipschitz constant of f over D be defined by

$$L(f) = \sup_{x, y \in D, x, \neq y} \|f(x) - f(y)\| / \|x - y\|, \tag{1.2}$$

where $\|\cdot\|$ is a given norm in R^n. Then it is known [1] that if $L(f) < 1$ and there exists an $x^{(0)} \in D$ such that

$$S = \{h;\ \|h - x^{(0)}\| \leqq L(f)\|x^{(1)} - x^{(0)}\|/(1 - L(f))\} \subset D, \tag{1.3}$$

then (1.1) has exactly one solution x^* in D and

$$\|x^{(1)} - x^*\| \leqq L(f)\,\|x^{(1)} - x^{(0)}\|/(1 - L(f)), \tag{1.4}$$

where $x^{(1)} = f(x^{(0)})$.

When $L(f)$ is finite, the Dahlquist constant of f over D (see [3]) is defined by

$$d(f) = \lim_{h \to +0} (L(I + hf) - 1)/h. \tag{1.5}$$

Söderlind [2] has shown that if x^* is a solution of (1.1), $x^{(0)}$, $x^{(1)} \in D$ and $d(f) < 1$, then

$$\|x^* - x^{(1)}\| \leqq L(f)\,\|x^{(1)} - x^{(0)}\|/(1 - d(f)), \tag{1.6}$$

where $x^{(1)} = f(x^{(0)})$. Since $d(f) \leqq L(f)$, the estimate (1.6) gives a smaller error bound than (1.4) when $d(f) < L(f)$ and especially when $d(f) < 0$.

For $x = (x_1, x_2, \dots, x_n)^T$ and $y = (y_1, y_2, \dots, y_n)^T \in C^n$, let $v(x) = (|x_1|, |x_2|, \dots, |x_n|)^T$ and write $x \geqq y$ if x_i and y_i are real and $x_i \geqq y_i$ $(i = 1, 2, \dots, n)$. Denote by $\sigma(A)$ the spectral radius of an $n \times n$ matrix A. For real $n \times n$ matrices $A = (a_{ij})$ and $B = (b_{ij})$ write $A \geqq B$ if $a_{ij} \geqq b_{ij}$ $(i, j = 1, 2, \dots, n)$. Then Urabe [5] has shown that if there exists a matrix $K \geqq 0$ such that