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Maximal ordered fields of rank n

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In this paper, the field of real numbers and the field of rational numbers will be denoted by $\mathbf{R}$ and $\mathbf{Q}$ respectively.

We denote an ordered field by (F, σ) or simply F, where σ is an ordering of a field F. For ordered fields (F, σ) and (K, τ), we say that K/F is an extension of ordered fields if K/F is an extension of fields and τ is an extension of σ. Let F be an ordered field. Then the following statements are equivalent:

(1) F is isomorphic to $\mathbf{R}$.

(2) F is a complete archimedean ordered field.

(3) F is archimedean and for any archimedean ordered field K, there is an order preserving isomorphism of K with a subfield of F.

(4) F is archimedean and for any proper extension of ordered fields K/F, K is not archimedean.

In §2, we define the *rank* of an ordered field. An ordered field F is archimedean if and only if F is of rank 0, and so $\mathbf{R}$ is characterized as a maximal ordered field of rank 0. For the case of rank n, $n \geqq 1$, we consider the following three conditions:

(1) F is a complete ordered field of rank n.

(2) F is of rank n and for any ordered field K of rank n, there is an order preserving isomorphism of K with a subfield of F.

(3) F is of rank n and for any proper extension of ordered fields K/F, rank $K > n$.

If $n \geqq 1$, the above three conditions are not equivalent. In fact (2) and (3) are equivalent, and (1) follows from (3) but not conversely. We say that F is a *maximal* ordered field of rank n if F satisfies the condition (3). The purpose of this paper is to show that there is a maximal ordered field of rank n and it is unique up to isomorphism.

§1. The rank of an ordered field

Let F be an ordered field and v be a valuation of F. The valuation ring, the group of units and the maximal ideal of v will be denoted by A, U and M respectively. Then the following statements are equivalent (cf. [5], Theorem 2.3):

(1) If $0 < a$ and $v(a) < v(b)$, then $b < a$.

(2) A is convex.