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Bessel capacity of symmetric generalized Cantor sets

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§1. Introduction

In [8] M. Ohtsuka obtained a necessary and sufficient condition for a symmetric generalized Cantor set to be of zero α-(or logarithmic-) capacity. In the non-linear potential theory the Bessel capacity of Cantor sets of special type was estimated in Maz'ya and Khavin [6]. In order to explain their results, let us recall the definitions of Bessel capacity and symmetric generalized Cantor sets.

Let $g_\alpha = g_\alpha(x)$ be the Bessel kernel of order α, $0<\alpha<\infty$, on the n-dimensional Euclidean space R^n $(n \geqq 1)$, whose Fourier transform is $(1+|\xi|^2)^{-\alpha/2}$. The Bessel capacity $B_{\alpha,p}$ is defined as follows: For a set $A \subset R^n$,

$$B_{\alpha,p}(A) = \inf \int f(x)^p dx,$$

the infimum being taken over all functions $f \in L_p^+$ such that

$$g_\alpha * f(x) \geqq 1 \qquad \text{for all} \quad x \in A.$$

We shall always assume that $1<p<\infty$ and $0<\alpha p \leqq n$.

Let $\{k_j\}_{j=1}^\infty$ be a sequence of integers and $\{\ell_j\}_{j=0}^\infty$ be a sequence of positive numbers such that $k_j \geqq 2$ and $k_{j+1}\ell_{j+1} < \ell_j$ $(j \geqq 0)$. Let $\delta_{j+1} = (\ell_j - k_{j+1}\ell_{j+1})/(k_{j+1}-1)$ $(j=0, 1,\dots)$. Let I be a closed interval of length ℓ_0 in R^1. In the first step, we remove from I (k_1-1) open intervals each of the same length δ_1 so that k_1 closed intervals $I_i^{(1)}$ $(i=1,\dots, k_1)$ each of length ℓ_1 remain. Set $E^{(1)} = \bigcup_{i=1}^{k_1} I_i^{(1)}$. Next in the second step, we remove from each $I_i^{(1)}$ (k_2-1) open intervals each of the same length δ_2 so that k_2 closed intervals $I_{i,j}^{(2)}$ $(j=1,\dots, k_2)$ each of length ℓ_2 remain. We set $E^{(2)} = \bigcup_{i=1}^{k_1} \bigcup_{j=1}^{k_2} I_{i,j}^{(2)}$. We continue this process and obtain $E^{(j)}$, $j \geqq 1$. We define $E = \bigcap_{j=1}^\infty E_n^{(j)}$, where the set $E_n^{(j)} = E^{(j)} \times \cdots \times E^{(j)}$ is the product set of n $E^{(j)}$'s in R^n. We call the set E the n-dimensional symmetric generalized Cantor set constructed by the system $[\{k_j\}_{j=1}^\infty, \{\ell_j\}_{j=0}^\infty]$.

The Cantor set E considered by Maz'ya and Khavin [6] is the one constructed as above with $k_j = 2$ for all $j \geqq 1$. For such a Cantor set E, they proved the following theorem.

THEOREM A. *If $\alpha p < n$, then*