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Genera and arithmetic genera of commutative rings

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Introduction

Let $(R, \mathfrak{m})$ be a d-dimensional noetherian local ring and I an $\mathfrak{m}$-primary ideal of R. Then it is well known that there exist (uniquely determined) integers e_i $(0 \leqq i \leqq d)$ such that

$$\ell(R/I^{n+1}) = e_0\binom{n+d}{d} - e_1\binom{n+d-1}{d-1} + \cdots + (-1)^d e_d$$

for all sufficiently large n. e_0 is the multiplicity of I. The significance of the other coefficients e_i has been studied by some people (cf. [4], [10], [16], [17], [19], [23]). But it seems that much more remains to be done. The aim of this paper is to study the significance of the invariants e_d and $e_0 - e_1 + \cdots + (-1)^d e_d$. We introduce the notion of *genus* and *arithmetic genus* of an ideal, and study the properties of local rings by these invariants. We also introduce the notion of *normal genus* and *normal arithmetic genus* by considering the function $\ell(R/\overline{I^{n+1}})$, where $\bar{J}$ denotes the integral closure of an ideal J. These invariants might be better to study singularities.

In §1, we investigate properties of general polynomial functions by introducing various invariants (e.g. the genus and the arithmetic genus) of a polynomial function.

In §2, we consider the Hilbert functions of graded modules. We express the genera by the local cohomology modules and study the relation of the genera and the vanishing of local cohomology modules.

In §3, the (normal) genus and the (normal) arithmetic genus of an ideal are defined, and we express them in terms of the sheaf cohomology of (normalized) blowing-up schemes.

In §4, we investigate the relation between the genera and the reduction exponents of ideals.

In §5 and §6, we examine the cases of dimension one and two (curve and surface singularities) in detail.

Notation and terminology

All rings are commutative noetherian rings with unit.