HIROSHIMA MATH. J.
17 (1987), 29–40

Normality, seminormality and quasinormality of $Z[\sqrt[n]{m}]$

Dedicated to Professor Masayoshi Nagata on his 60th birthday

Hiroshi TANIMOTO

(Received March 11, 1986)

Introduction

In [11], Ooishi gave a necessary and sufficient condition for $\boldsymbol{Z}[\sqrt{m}]$ to be seminormal. In this paper, we will study $\boldsymbol{Z}[\sqrt[n]{m}]$ and give the criteria for $\boldsymbol{Z}[\sqrt[n]{m}]$ to be normal, p-seminormal, seminormal and quasinormal. First, we treat the normality of $\boldsymbol{Z}[\sqrt[n]{m}]$. Next, we construct some elements which are integral over $\boldsymbol{Z}$. Then using these elements, we study the p-seminormality, the seminormality and the quasinormality of $\boldsymbol{Z}[\sqrt[n]{m}]$.

The writer heartily thanks Prof. H. Matsumura who gave him continuous encouragement.

§ 1. Notation, terminology and preliminary results

Let A be a noetherian reduced ring. If the canonical homomorphism $\mathrm{Pic}\, A \to \mathrm{Pic}\, A[X]$ (or $\mathrm{Pic}\, A \to \mathrm{Pic}\, A[X, X^{-1}]$) is an isomorphism, where X is a variable, A is said to be *seminormal* (or *quasinormal*, resp.), and for an integer p if the kernel of $\mathrm{Pic}\, A[X] \to \mathrm{Pic}\, A$ has no p-torsion, A is said to be *p-seminormal.* These are chracterized as follows. The seminormality (or the p-seminormality) of A is equivalent to that if $x \in Q(A)$ satisfies x^2, $x^3 \in A$ (or x^2, x^3, $px \in A$, resp.), then $x \in A$ (cf. [5] or [12]). On the other hand in the case that $\dim A = 1$ and A is a domain, A is quasinormal if and only if the following conditions are satisfied: (1) A is seminormal, and (2) if $x \in Q(A)$ satisfies $x^2 - x$, $x^3 - x^2 \in A$, then $x \in A$ (cf. [10]). These are our main tools in this paper. Now normality, seminormality and p-seminormality are local properties, that is, A is normal (or seminormal, p-seminormal) if and only if so is $A_{\mathfrak{m}}$ for all maximal ideals $\mathfrak{m}$ of A (cf. [12]). If $\dim A = 1$ and A is a domain, quasinormality is a local property (cf. [2]). For an ideal $\mathfrak{a}$ of A, we write $V(\mathfrak{a}) = \{\mathfrak{p} \in \mathrm{Spec}\, A \mid \mathfrak{a} \subseteqq \mathfrak{p}\}$. And we denote the normalization of A in $Q(A)$ by $\tilde{A}$.

We denote the set of natural numbers by $\boldsymbol{N}$, the set of integers by $\boldsymbol{Z}$, the set of rational numbers by $\boldsymbol{Q}$ and the prime field of characteristic p by $\boldsymbol{F}_p$.

Throughout this paper, m and n are integers with $n \geqq 2$. Moreover when $X^n - m$ is an irreducible polynomial over $\boldsymbol{Z}$, we denote a root of $X^n - m = 0$ by $\sqrt[n]{m}$.