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Cuts of ordered fields

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We denote an ordered field by (F, σ) or simply F, where σ is an ordering of a field F. For ordered fields (F, σ) and (K, τ), we say that K/F is *an extension of ordered fields* if K/F is an extension of fields and τ is an extension of σ. In this paper, $F(x)$ always means a simple transcendental extension of F. A pair (C, D) of subsets of F is called a cut of F if $C \cup D = F$ and $c < d$ for any $c \in C$ and $d \in D$. Let $(F(x), \tau)/(F, \sigma)$ be an extension of ordered fields. Then $g(\tau) := (C, D)$, where $C = \{a \in F;\ a < x\}$ and $D = \{a \in F;\ a > x\}$, is a cut of F. If F is a real closed field, then g is a bijective map from the set of all orderings of $F(x)$ to the set of all cuts of F (Theorem 1.2). In [2], we defined the *rank* of an ordered field and we said that an ordered field F is *a maximal ordered field of rank n* if rank $F = n$ and for any proper extension K/F of ordered fields, rank $K > n$.

Let F be a real closed field of finite rank n and let $A_1 \subset \cdots \subset A_n \subset A_{n+1} = F$ be the compatible valuation rings of F. In this paper, we define the subsets W_i, $i = 1, \dots, n+1$, of the set of all cuts of F (Definition 3.4) and show that for an ordering τ of $F(x)$, the following statements are equivalent (Theorem 3.10):

(1) $g(\tau) \in W_i$.

(2) There exist distinct convex valuation rings B and B' of $F(x)$ with respect to τ such that $B \cap F = B' \cap F = A_i$.

As a corollary of the above assertion, we have the following statement: $\operatorname{rank}(F(x), \tau) = \operatorname{rank} F + 1$ if and only if $g(\tau) \in \bigcup_{i=1}^{n+1} W_i$. In particular, F is a maximal ordered field if and only if any cut of F is contained in $\bigcup_{i=1}^{n+1} W_i$.

§1. Real closed fields and cuts

Let F be an ordered field. If C and D are subsets of F, we write $C < D$ if $c < d$ for all $c \in C$, $d \in D$. If $a \in F$, then we write $C < a$ or $a < D$ instead of $C < \{a\}$ or $\{a\} < D$, respectively. A pair (C, D) of subsets of F is called a cut of F if $F = C \cup D$ and $C < D$. We regard (F, ϕ) and (ϕ, F) as cuts of F. Throughout this paper, we denote by X the set of orderings σ of $F(x)$ where $(F(x), \sigma)/F$ is an extension of ordered fields. Let C_F be the set of all cuts of F. We define the map $g_F\colon X \to C_F$ by $g_F(\sigma) = (C, D)$, where $C = \{c \in F;\ c < x(\sigma)\}$ and $D = \{d \in F;\ x < d(\sigma)\}$; here we write $a < b(\sigma)$ if $a < b$ with respect to the ordering σ. It is well known that there is an ordering $\sigma \in X$ such that $F < x(\sigma)$ and it is uniquely determined (cf. [1]). In this case, it is clear that $g_F(\sigma) = (F, \phi)$.