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Minimal conditions for Lie algebras and finiteness of their dimensions

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Introduction

We shall be concerned with Lie algebras which are not necessarily finite-dimensional over an arbitrary field $\mathfrak{k}$ unless otherwise specified, and mostly follow [4] for the use of notations and terminology. The classes Min-$\lhd^{\alpha}$ (α is an ordinal), Min-si, Min-asc, Min-ser, Min and $\mathfrak{F}$ are related by the series of inclusions

$$\text{Min-}\lhd \geq \text{Min-}\lhd^2 \geq \text{Min-}\lhd^3 \geq \cdots \geq \text{Min-si} \geq \text{Min-}\lhd^{\omega} \geq \text{Min-}\lhd^{\omega+1}$$
$$\geq \cdots \geq \text{Min-asc} \geq \text{Min-ser} \geq \text{Min} \geq \mathfrak{F},$$

where Min-ser denotes the class of Lie algebras satisfying the minimal condition for serial subalgebras. Concerning these inclusions, Amayo and Stewart have proved that if $\mathfrak{k}$ is of characteristic zero, then Min-$\lhd >$ Min-$\lhd^2 =$ Min-si (cf. [4, Theorem 8.1.4]), and that if $\mathfrak{k}$ is of characteristic $p>0$, then Min-$\lhd^2 >$ Min-$\lhd^3 =$ Min-si (cf. [4, Proposition 8.1.5 and the example in §8.3]). Furthermore, Stewart has proved that Min-si = Min-asc ([13, Theorem]), and that if $\mathfrak{k}$ is of characteristic zero, then $\mathrm{L}\mathfrak{F} \cap \mathrm{Min} = \mathfrak{F}$ (cf. [4, Corollary 10.2.2]). The first purpose of this paper is to investigate the relationship among the classes Min-si, Min-ser and Min. The second one is to present sufficient conditions for Lie algebras satisfying minimal conditions to be finite-dimensional.

In Section 1 we shall first prove that if $\mathfrak{k}$ is of characteristic zero, then $\mathrm{L}\mathfrak{F} \cap$ Min-$\lhd^2 = \mathrm{L}\mathfrak{F} \cap$ Min-ser (Corollary 1.6), and secondly prove that $\mathrm{L}\mathfrak{F} \cap$ Min-ser $>$ $\mathrm{L}\mathfrak{F} \cap$ Min and so Min-ser $>$ Min (Theorem 1.7). In consequence of these results, we shall conclude that if $\mathfrak{k}$ is of characteristic zero, then $\mathrm{L}\mathfrak{F} \cap$ Min-$\lhd > \mathrm{L}\mathfrak{F} \cap$ Min-$\lhd^2 = \mathrm{L}\mathfrak{F} \cap$ Min-ser $> \mathrm{L}\mathfrak{F} \cap \mathrm{Min} = \mathfrak{F}$.

In Section 2 we shall prove that $\{\mathrm{J(asc)}, \hat{\mathrm{E}}\mathrm{(asc)}\}\mathfrak{I} \cap$ Min-si $= \{\mathrm{J(ser)}, \hat{\mathrm{E}}\}\mathfrak{I}^* \cap$ Min-ser $= \mathfrak{F}$, where $\mathfrak{I}$ (resp. $\mathfrak{I}^*$) denotes the class of Lie algebras having no infinite-dimensional, simple (resp. absolutely simple) factors of ideals (Theorem 2.5). Especially, if $\mathfrak{k}$ is of characteristic zero, then $\mathrm{L(ser)}\mathfrak{F} \leq \mathfrak{I}$ and so $\mathrm{L(ser)}\mathfrak{F} \cap$ Min-$\lhd^2 = \mathfrak{F}$ (Corollary 2.6).

In Section 3 we shall present classes of generalized soluble Lie algebras for which minimal conditions imply finiteness of their dimensions. For example, we shall prove that if $\mathfrak{X}$ is an $\{\mathrm{I}, \mathrm{Q}\}$-closed class of Lie algebras such that every