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# Nonlinear equations on a Lie group

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## Introduction

In their paper [11, 12], I. Hauser and F. J. Ernst proved in the affirmative the Geroch conjecture [2, 3] that all stationary axisymmetric solutions of the Einstein field equations can be essentially derived from the Minkowski space metric by means of Kinnersley-Chitre transformations. In the subsequent paper [13], they extended this result to the case of $N$ Abelian gauge fields interacting with gravitation which contains as special cases the Einstein field equations ($N=0$) and the Einstein-Maxwell field equations ($N=1$). We call their equations the Hauser-Ernst equations.

The purpose of this paper is to give a slightly different formulation of the Hauser-Ernst equations in such a way that the theory of real semisimple Lie groups can be easily applied to it; replacing the real simple group $SU(N+1, 1)$ in their formulation by more general real simple Lie groups, we shall prove our version of the "generalized Geroch conjecture". Our formulation is based on the framework of the theory of homogeneous spaces of Lie groups. The most striking is the analogy with concepts of finite (or infinite) dimensional generalized flag manifolds, parabolic subgroups, the Bruhat (or rather the Birkhoff) decomposition and so on (cf. [14]). In particular we give a new proof, using the Birkhoff decomposition in place of the homogeneous Hilbert problem.

The paper is organized as follows.

In Section 1 we introduce a certain topology into a group of analytic loops on a Lie group; provided with this topology the analytic loop group becomes a topological group. Theorem 1.6 implies that for an arbitrary linear algebraic group any loop sufficiently near the unit loop has a Birkhoff decomposition.

In Section 2 we follow the idea of Hauser-Ernst to define the action of the analytic loop group.

In Section 3, with the aid of a certain involutive real automorphism of $GL(N, \boldsymbol{C})$, we put the Hauser-Ernst equations into simple formulas, from which we can easily deduce various group theoretical properties of solutions. Theorem 3.2 describes the dependency of solutions on their axial values.

Finally, in Section 4 we start with constructing a "seed solution" which plays the same role as the Minkowski space metric in the theory of Hauser-Ernst. Theorem 4.6 assures that our loop group generates solutions. The results of