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Closure relations for orbits on affine symmetric spaces under the action of parabolic subgroups. Intersections of associated orbits

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§ 1. Introduction

Let G be a connected Lie group, σ an involution of G and H an open subgroup of $G^\sigma=\{x\in G\mid \sigma x=x\}$. Then the G-homogeneous manifold $H\backslash G$ is called an affine symmetric space. Suppose that G is a real semisimple Lie group. Let P be a minimal parabolic subgroup of G and P' a parabolic subgroup of G containing P. Then the double coset decomposition $H\backslash G/P$ is studied in [2], and [5], the relation between $H\backslash G/P'$ and $H\backslash G/P$ is studied in [3], and the closure relation for $H\backslash G/P$ is studied in [4].

Let θ be a Cartan involution of G such that $\sigma\theta=\theta\sigma$. Put $K=G^\theta$ and let H^{a} be the open subgroup of $G^{\sigma\theta}$ such that $K\cap H=K\cap H^{\mathrm{a}}$. Then $H^{\mathrm{a}}\backslash G$ is called the affine symmetric space associated to $H\backslash G$. Let A be a θ-stable split component of P and put $U=\{x\in K\mid xAx^{-1}$ is σ-stable$\}$.

There exists a natural one-to-one correspondence between the double coset decompositions $H\backslash G/P'$ and $H^{\mathrm{a}}\backslash G/P'$ given by $D\rightarrow D^{\mathrm{a}}=H^{\mathrm{a}}(D\cap U)P'$ for $H-P'$ double cosets D in G ([2], [3]). Moreover it follows easily from Corollary of Theorem in [4] that this correspondence reverses the closure relations for the double coset decompositions. In this paper we prove the following theorem.

THEOREM. *Let D_1 and D_2 be arbitrary $H-P'$ double cosets in G. Then we have the following.*

(i) $D_1^{\mathrm{cl}}\supset D_2\Leftrightarrow D_1\cap D_2^{\mathrm{a}}\neq\emptyset$.

(ii) *Let $I(D_1, D_2)$ be the set of all the $H-P'$ double cosets D in G such that $D_1^{\mathrm{cl}}\supset D^{\mathrm{cl}}\supset D_2$. Then*

$$(D_1\cap D_2^{\mathrm{a}})^{\mathrm{cl}}\cap D_2^{\mathrm{a}}=\bigcup_{D\in I(D_1,D_2)}D\cap D_2^{\mathrm{a}}.$$

(iii) *Let x be an element of U. Then $HxP'\cap H^{\mathrm{a}}xP'=(K\cap H)xP'$.*

(iv) *$D_1\cap D_2^{\mathrm{a}}$ is nonempty and closed in $G\Leftrightarrow D_1=D_2$.*

Example. Let G_1 be a connected semisimple Lie group, θ_1 a Cartan involution of G_1, $K_1=\{x\in G_1\mid \theta_1x=x\}$, and P_1 a minimal parabolic subgroup of G_1 with a θ_1-stable split component A_1. Let P_1' and P_1'' be parabolic subgroups of