HIROSHIMA MATH. J.
18 (1988), 399–412

Time change and orbit equivalence in ergodic theory

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(Received September 16, 1987)

§ 1. Introduction

The theory of the orbit equivalence has occupied an important place in ergodic theory and is closely related to the notion of the time changes. In this paper we shall investigate the relation between them.

In §2 and §3 we shall follow Totoki's paper [6] with some additional results. He constructed time changes from given flows by means of so called additive cocycles, which have the similar properties to those of the lags of the parameters in the orbit equivalence. To speak in more detail, an additive cocycle φ with respect to a flow $(X, \mathscr{B}, \mu, \{T_t\})$ (usually on a Lebesgue probability space) is a measurable map from $\mathbf{R} \times X$ to $\mathbf{R}$, which satisfies the equation $\varphi(t+s, x) = \varphi(s, x) + \varphi(t, T_s x)$ $t, s \in \mathbf{R}$, $x \in X$, with $\varphi^x(t) \equiv \varphi(t, x)$ non-decreasing (not necessarily strictly increasing) and continuous in t for x in a $\{T_t\}$-invariant co-null set. Among them additive cocycles of the forms $\varphi(t, x) = \int_0^t f(T_s x) ds$ with non-negative f's are important ones, as we shall see later on. For an additive cocycle φ, we shall see $\lim_{h \to 0} \varphi(h, x)/h$ exists a.e. by the same method as in the Wiener's local ergodic theorem, from which we have the Lebesgue decomposition of φ. The time change is defined as the same way as in [6] and its invariant measure is described by φ. Especially for S-flows, we have S-representations of time changes.

In §4 we shall argue the relation between these time changes and the orbit equivalence. Time change defines an equivalence relation, which is denoted by $\sim$, among ergodic flows. That is, $\{S_t\} \sim \{T_t\}$ if there exists some integrable additive cocycle φ such that the time change $\{S_t^\varphi\}$ is isomorphic to $\{T_t\}$. It turns out that $\sim$ is an equivalence relation and we shall show, as expected naturally, that $\sim$ and the orbit equivalence $\overset{\mathcal{Q}}{\sim}$ coincide. This is the main theorem in this paper.

In §5 we shall treat a special problem of isomorphic relation. The time change of an S-flow by an additive cocycle φ is isomorphic to those by φ_n, where φ_n's are the ones defined by the functions f_n's such that $f_n \to f = \lim_{h \to 0} \varphi(h, \cdot)/h$ a.e. In this connection we have the following problem. Is the time change by φ_{ac} isomorphic to the one by φ, where φ_{ac} is the additive cocycle defined by f? Generally this is not true except for the tiivial case when φ is of integral form (defined by f).