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Liapunov functions and boundedness for differential and delay equations

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1. Introduction

In the theory of Liapunov's direct method for a system of ordinary differential equations

$$x' = H(t, x) \tag{1}$$

where $H: [0, \infty) \times R^n \to R^n$ is continuous, in order to prove that all solutions tend to zero as $t \to \infty$ it suffices to find a continuous function $V: [0, \infty) \times R^n \to [0, \infty)$ and continuous functions $W_i: [0, \infty) \to [0, \infty)$, $W_i(0) = 0$, $W_i(r)$ increasing, such that

(i) $W_1(|x|) \leq V(t, x) \leq W_2(|x|)$,
(ii) $V'_{(1)}(t, x) \leq -W_3(|x|)$,

and

(iii) $W_1(r) \to \infty$ as $r \to \infty$.

Corduneanu [2] showed that if

(iv) $V'_{(1)}(t, x) \leq -h(t, V)$

and if the solutions of $\{r' = h(t, r),\ r(t_0) = V(t_0, x_0)\}$ tend to zero, then $V(t, x(t)) \to 0$ as $t \to \infty$.

Three problems have persisted in fundamental applications: (ii), (iii), and (iv) fail. The typical example is the scalar system

$$x' = y$$
$$y' = -q(x, y)y - g(x)$$

where $q(x, y) \geq 0$, $xg(x) > 0$ if $x \neq 0$. The function

$$V(x, y) = y^2 + 2\int_0^x g(s)ds$$

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