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Any group is represented by an outerautomorphism group

Dedicated to Professor Hirosi Toda on his 60th birthday

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1. Introduction

Recently S. Kojima showed in [5] that any finite group G is isomorphic to the outerautomorphism (class) group $\mathrm{Out}(\pi) = \mathrm{Aut}(\pi)/\mathrm{Inn}(\pi)$ of some discrete subgroup π of $PSL(2, \boldsymbol{C})$ (cf. § 5). The purpose of this paper is to show the following theorem.

THEOREM. *For any group G there is a group π such that G is isomorphic to the outerautomorphism group* $\mathrm{Out}(\pi)$ *of* π.

In terms of the homotopy theory our theorem says that G is isomorphic to the group $\mathscr{E}(K(\pi, 1))$ of free homotopy classes of homotopy self-equivalences, with multiplication by the composition, of the Eilenberg-MacLane space $K(\pi, 1)$ (Corollary 4.2). In [3] J. de Groot showed that any group G is isomorphic to the homeomorphism group $\mathrm{Homeo}(X)$ of some metric space X. This implies also that G is isomorphic to the (outer)automorphism group of the ring of real-valued continuous functions on X. Moreover, we can see that $\mathrm{Homeo}(X) = \mathscr{E}(X)$ in his specific example (cf. § 4).

It is easy to see that there is no group π whose automorphism group $\mathrm{Aut}(\pi)$ is the cyclic group of odd order $\neq 1$ (cf. [3]). In contrast with this we may ask if there is any based space X such that G is isomorphic to the group $\mathscr{E}_0(X)$ of based homotopy classes of based homotopy self-equivalences of X. In [9] S. Oka showed that $\mathscr{E}_0(X_0)$ is a cyclic group of order n for some 1-connected finite CW complex X_0 unless $n \equiv 8 \bmod 16$.

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