HIROSHIMA MATH. J.
19 (1989), 141–166

Integration in mixed topological spaces*)

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(Received January 20, 1988)

In the present paper we are concerned with integration of functions with values in mixed topological spaces. The theory of Lebesgue integral on a general measure space has been extended to the case of functions taking their values in Banach spaces by Birkhoff [1], Bochner [3], Pettis [17] and others [12]. These vector integration theories have been extended further to the case of locally convex spaces by Phillips [18] and Rickart [20]. Mixed topological spaces form an important class of locally convex spaces. These spaces have many interesting properties and are very abundant. The mixed topological structures often appear in various problems from analysis as well as the theory of partial differential equations, and it is expected that the theory of integration in mixed topological spaces is not only significant from the theoretical point of view, but also it has considerable practical applicability.

A mixed topological space is a locally convex space (E, τ) equipped with a bornology on E. A subset B of E is called a ball in E if it is an absolutely convex subset which does not contain a nontrivial subspace. By a bornology on E we mean a family $\mathscr{B}$ of balls in E with the four properties below: (a) $\mathscr{B}$ is a covering of E, (b) $\lambda B \in \mathscr{B}$ for $B \in \mathscr{B}$ and $\lambda > 0$, (c) for $B, C \in \mathscr{B}$ there exists $D \in \mathscr{B}$ with $B + C \subset D$, and (d) if $B \in \mathscr{B}$ and C is a ball contained in B then $C \in \mathscr{B}$. If in particular there exists a countable subfamily $\{B_n\}$ of $\mathscr{B}$ such that any element $B \in \mathscr{B}$ is contained in some B_n, then $\mathscr{B}$ is said to be of countable type. To the locally convex space (E, τ) there corresponds a bornology $\mathscr{B}_\tau$ called the von Neumann bornology on E that is the family of all τ-bounded, absolutely convex subsets of E. In this paper we restrict ourselves to a bornology $\mathscr{B}$ on E satisfying the compatibility condition

$$\mathscr{B} \subset \mathscr{B}_\tau ,$$

and assume that there exists a countable subfamily $\{B_n\}$ of $\mathscr{B}$ such that any element $B \in \mathscr{B}$ is contained in some B_n and any B_n is τ-closed. Now to the triplet $(E, \mathscr{B}, \tau)$ one can introduce a new locally convex topology that is finer than the original topology τ and denote it by $\gamma \equiv \gamma[\mathscr{B}, \tau]$. This topology γ is

*) This research is partially supported by Grant-in-Aid for Scientific Research, Ministry of Science and Culture, Japan.