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The weak supersolution-subsolution method for second order quasilinear elliptic equations

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1. Introduction

This paper is concerned with the Dirichlet problem for second order quasilinear elliptic equations of the type

$$-\operatorname{div} A(x, \nabla u) + B(x, u, \nabla u) = 0 \qquad \text{in } \Omega\,, \tag{1.1}$$

$$u = g \qquad \text{on } \partial\Omega\,, \tag{1.2}$$

where Ω is either a bounded domain or an exterior domain in $\boldsymbol{R}^N$, A is a given N-vector function of the variables x and $\nabla u = (\partial u/\partial x_1, \dots, \partial u/\partial x_N)$, B is a given scalar function of the variables x, u and ∇u, and g is a function given on the boundary $\partial\Omega$ of Ω. We allow the domain Ω to be the entire space $\boldsymbol{R}^N$, in which case the boundary condition (1.2) is void. Equation (1.1) is allowed to be degenerate so that the nonlinear pseudo-Laplacian equation

$$-\operatorname{div}(|\nabla u|^{p-2}\nabla u) + B(x, u, \nabla u) = 0 \qquad \text{in } \Omega\,, \qquad p > 1\,, \tag{1.3}$$

is included as a special case of it. Our objective here is to develop the method of supersolutions and subsolutions for constructing weak solutions of the problem (1.1)–(1.2) and for analyzing the structure of the set of weak solutions thus constructed.

A systematic study of nonlinear elliptic boundary problems by means of the supersolution-subsolution method was initiated by Nagumo [21], who considered the semilinear equation

$$-\textstyle\sum_{i,j=1}^N a_{ij}(x)\dfrac{\partial^2 u}{\partial x_i\,\partial x_j} + B(x, u, \nabla u) = 0 \tag{1.4}$$

in a bounded domain Ω and established an existence theorem asserting that the problem (1.4)–(1.2) has a classical solution if suitable classical quasi-supersolutions and quasi-subsolutions are known to exist. (By a quasi-supersolution (quasi-subsolution) we mean a function which is expressed locally as the minimum (maximum) of a finite number of supersolutions (subsolutions) of the problem.) Nagumo's existence theory has been generalized and extended in various directions. Among other things Akô [1] (see also Hirai and Akô [14])