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Huygens property of parabolic functions and a uniqueness theorem

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It is well known that the zero function is the only non-negative parabolic function (i.e., solution of the heat equation) on $\boldsymbol{R}^n \times (0, T)$ which vanishes continuously on $\boldsymbol{R}^n \times \{0\}$ (cf. [2, Part 1, Chap. XVI], [6, Chap. VIII]). In this paper, considering the Huygens property of minimal parabolic functions (see Proposition 2), we shall show the following generalization of the above result.

THEOREM. *Let Ω be a Lipschitz cone in $\boldsymbol{R}^n$ $(n \geq 1)$ and let $0 < T \leq \infty$. If u is a non-negative parabolic function in the cylinder $\Omega \times (0, T)$ and vanishes continuously on the parabolic boundary $\partial\Omega \times [0, T] \cup \Omega \times \{0\}$, then $u \equiv 0$.*

Here, we say that a domain Ω is a *Lipschitz cone* if there exists a domain E in the unit sphere S^{n-1} such that $\Omega = \{x \neq 0;\ x/\|x\| \in E\}$ and $\Omega \cap \{x;\ \|x\| < 1\}$ is a Lipschitz domain, where $\|x\|$ denotes the euclidean norm of x in $\boldsymbol{R}^n$.

In Section 5, we remark that the above assertion is also valid for solutions of parabolic equations and for a slightly more general domain Ω.

§1. The Huygens property

In this section $D = \Omega \times (0, T)$ will be a cylinder with a domain Ω in $\boldsymbol{R}^n$ $(n \geq 1)$ and $0 < T \leq \infty$.

A solution of the heat equation on D is said to be *parabolic* on D. We denote by $H^+(D)$ the set of all non-negative parabolic functions on D. Also we denote by $\partial_p D$ the parabolic boundary of D, i.e., $\partial_p D = \partial\Omega \times [0, T) \cup \Omega \times \{0\}$, and by G_D the Green function of D with respect to the heat equation (cf. [2, p.298]).

For a non-negative function u and $0 < s < T$, we define a function u_s by

$$u_s(x, t) = \begin{cases} u(x, t) & \text{on} \quad \Omega \times (0, s] \\ \displaystyle\int_\Omega G_D((x, t), (y, s))\, u(y, s)\, dy & \text{on} \quad \Omega \times (s, T). \end{cases}$$

Then we have

LEMMA 1. *For $u \in H^+(D)$ and $0 < s < T$, $u_s \in H^+(D)$ and $u_s \leq u$ on D.*