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On the products $\beta_s\beta_t$ in the stable homotopy groups of spheres

Dedicated to Professor Shôrô Araki on his sixtieth birthday

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§1. Introduction

For a prime $p \geqq 5$, H. Toda [7] introduced the β-family $\{\beta_s | s \geqq 1\}$ and showed the relation $uv\beta_s\beta_t = st\beta_u\beta_v$ $(s + t = u + v)$ in the p-component of the stable homotopy groups $\pi_*(S)$ of spheres. An easy consequence of this relation is $\beta_s\beta_t = 0$ if $p | st$, since the order of β_s is p. In this paper we find the following

THEOREM 1.1. *Let s and t be positive integers with $p \nmid st$. Then,*

$$\beta_s\beta_t \neq 0 \ \textit{in} \ \pi_*(S) \quad \text{if } s + t \in I,$$

where $I = \{kp^i - (p^{i-1} - 1)/(p - 1) | i \geqq 1, \ p \nmid k + 1\}$.

Consider the Adams-Novikov spectral sequence converging to $\pi_*(S)$, in which Miller, Revenel, and Wilson [1] defined the β-elements β_s $(s \geqq 1)$ surviving to β_s in $\pi_*(S)$. This sequence has sparsity in its E_2-term enough not to kill the product $\beta_s\beta_t$. Therefore the above theorem follows from the non-triviality in the following

THEOREM 1.2. *Let s and t be positive integers with $p \nmid st$. Then, in the E_2-term of the Adams-Novikov spectral sequence,*

$$\beta_s\beta_t \neq 0 \ \textit{if} \ s + t \in I.$$

Furthermore suppose that $s + t \geqq p^2 + p + 2$. Then we have

$$\beta_s\beta_t = 0 \ \textit{if} \ p \nmid (s + t)(s + t + 1), \ \textit{or if} \ s + t + 1 = kp \ \textit{and} \ p \nmid k(k + 1).$$

Notice that $p | n(n + 1)$ if $n \in I$. We also note that the relation "$\beta_s\beta_t = 0$ if $p | st$" is also valid in the E_2-term ([2], [6; Cor. 2.8]), and that $\beta_s\beta_t = 0$ if and only if $p | st$ in both $\pi_*(S)$ and the E_2-term for the case when $p = 5$ and $s + t \leqq p^2 - p + 1$ ([3; Chap. 7]).

This theorem does not determine whether or not $\beta_s\beta_t$ $(p \nmid st)$ is trivial in the E_2-term for the following cases:

a) $p^2 | s + t + p$, b) $s + t = kp^3 - p^2 - 1$, and c) $s + t = kp^2 - p - 1 \notin I$.

In §2, we recall the Brown-Peterson spectrum BP at p and the E_2-term