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Three Riemannian metrics on the moduli space of 1-instantons over CP^2

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1. Introduction

The natural metric on 5-sphere of radius 1 induces the Fubini-Study metric g_{FS} on the complex projective plane CP^2. The moduli space $\mathscr{M}$ of 1-instantons over (CP^2, g_{FS}) is homeomorphic to the cone on CP^2 (Buchdahl [B] and Furuta [F]). The generic part $\mathscr{M}^*$ of the moduli space carries three natural Riemannian metrics g_J (J = I, II and I–II). We refer to Matumoto [M] for the definition of the Riemannian symmetric tensors. In this paper we will give explicit formulas of the metrics and study their basic geometric properties.

Buchdahl and Furuta defined an $SU(3)$-equivariant diffeomorphism F: $CP^2 \times (0, 1) \cong \mathscr{M}^* = \mathscr{M}$-{cone point}. We use a local coordinate system $\mathbf{C}^2 \times (0, 1) \to CP^2 \times (0, 1)$ defined by $(W_1, W_2, \lambda) \to ([1, W_1, W_2], \lambda)$ with $W_1 = X_1 + iX_2$ and $W_2 = X_3 + iX_4$. Note that $F(\mathbf{C}^2 \times (0, 1))$ is open and dense in $\mathscr{M}^*$. The metric tensors split with respect to this coordinate system as

$$F^*g_J = \varphi_J(\lambda)\, d\lambda^2 + \psi_J(\lambda) g_{FS} \quad (\mathrm{J} = \mathrm{I}, \mathrm{II} \text{ and } \mathrm{I{-}II}).$$

More explicitly, we can write $\varphi_J(\lambda)$ and $\psi_J(\lambda)$ by using a new paramater $Z = 1 - \lambda^2$ as follows:

$$\varphi_I(\lambda) = 8\pi^2(Z^2 \log Z + 3Z \log Z - 3Z^2 + 2Z + 1)/Z(1 - Z)^3,$$

$$\psi_I(\lambda) = 4\pi^2(-6Z^2 \log Z + Z^3 + 6Z^2 - 9Z + 2)/(Z + 2)(1 - Z)^2;$$

$$\varphi_{II}(\lambda) = 16\pi^2(Z^2 - 2Z + 6)/15Z^2, \quad \psi_{II}(\lambda) = 8\pi^2(-3Z^2 - 4Z + 12)(1 - Z)/15Z;$$

$$\varphi_{I-II}(\lambda) = \varphi_{II}(\lambda), \quad \psi_{I-II}(\lambda) = 24\pi^2(Z^4 - Z^3 + 2Z^2 + 8)(1 - Z)/5Z(Z + 2)^2.$$

In fact, $\varphi_J(\lambda)$ and $\psi_J(\lambda)$ are positive for $0 < \lambda < 1$ and g_J defines actually the positive definite Riemannian metrics for not only J = I but also J = II and I–II.

From the above formulas or their asymptotic ones given in §4, we get the following proposition, where $K_J(u, v)$ (J = I, II and I–II) denote the sectional curvatures of F^*g_J.

PROPOSITION. (a) *As* $\lambda \to 1$ (*near the collar*) *all the sectional curvatures converge to the negative constant* $-5/32\pi^2$ *for* $(\mathscr{M}^*, g_{II})$ *and* $(\mathscr{M}^*, g_{I-II})$. *On* $(\mathscr{M}^*, g_I)$, *we can induce a* C^∞ *metric on* $\partial\mathscr{M}$ *so that* $(\partial\bar{\mathscr{M}}, g_I)$ *is isometric to*