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The injective hull of homotopy types with respect to generalized homology functors

Dedicated to Professor Masahiro Sugawara
on his sixtieth birthday

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A. K. Bousfield [1], [2] introduced the notion of the localization of spaces and spectra with respect to homology. In this note, we introduce the notion of the injective hull of spaces and spectra with respect to homology. We also prove that the class $A(Ho^s)$ of Bousfield equivalence classes of spectra [2] becomes a set i.e., has a cardinality.

1. Statement of results

$\mathscr{C}$, $\mathscr{S}$, $\tilde{\mathscr{C}}$, $\tilde{\mathscr{S}}$ denote the categories of CW-complexes, CW-spectra, and their homotopy categories respectively.

DEFINITION 1. Let $\mathscr{A}$, $\mathscr{B}$ be categories and $\mathscr{F}: \mathscr{A} \to \mathscr{B}$ a functor.

i) $A \in \mathrm{Ob}(\mathscr{A})$ is *$\mathscr{F}$-local* (resp. *$\mathscr{F}$-injective*) if for any B, $C \in \mathrm{Ob}(\mathscr{A})$ and any $f: B \to C$ with $\mathscr{F}(f)$: iso (resp. mono), $f^*: \mathscr{A}(C, A) \to \mathscr{A}(B, A)$ is an iso (resp. epi).

ii) A map $f: A \to B$ is an *$\mathscr{F}$-localization map* of A if B is $\mathscr{F}$-local and $\mathscr{F}(f)$ is an iso.

iii) A map $f: A \to B$ is an *$\mathscr{F}$-injective enveloping map* of A if f satisfies the following two conditions:

a) B is $\mathscr{F}$-injective and $\mathscr{F}(f)$ is a mono.

b) For any $C \in \mathrm{Ob}(\mathscr{A})$ and any $g: B \to C$, $\mathscr{F}(g)$ is monic if $\mathscr{F}(g \circ f)$ is monic.

iv) B is an *$\mathscr{F}$-injective hull* of A if there is an $\mathscr{F}$-injective enveloping map $f: A \to B$ of A.

Then we can prove the following:

THEOREM 1. *Let $h = (h_n | n \in \mathbf{Z}): \mathscr{D} \to \mathscr{G}r\mathscr{A}b$ ($\mathscr{D} \in \{\tilde{\mathscr{C}}, \tilde{\mathscr{S}}\}$) be a generalized homology functor which is representable by a spectrum where $\mathscr{G}r\mathscr{A}b$ is the category of $\mathbf{Z}$-graded abelian groups. Then it follows that*:

i) *Any object $A \in \mathrm{Ob}(\mathscr{D})$ has an h-injective enveloping map.*

ii) *Let $f: A \to B$ and $g: A \to C$ be h-injective enveloping maps of A. then there exists a map $k: B \to C$ such that $k \circ f = g$. Moreover, such a k is always an*