HIROSHIMA MATH. J.
**19** (1989), 515–520

# On renormings of nonreflexive Banach spaces with preduals

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(Received September 19, 1988)

## Introduction

Let $X$ be a Banach space. Let $X^*$ and $X^{**}$ denote the first and second dual spaces of $X$. We denote by $\pi$ the canonical map of $X$ into $X^{**}$. In what follows, $X$ will be identified with $\pi(X)$. It is well known that if a Banach space $X$ is isometric to a dual Banach space then there exists a projection $P$ with norm 1 from its second dual $X^{**}$ onto $X$. Davis and Johnson showed in [3] that every nonreflexive Banach space can be equivalently renormed in such a way that the renormed space is not isometric to a dual space. Dulst and Singer [4] then proved that this conclusion can be improved in the following form: Every nonreflexive Banach space $X$ admits an equivalent norm $\|\cdot\|$ such that for each projection $P$: $X^{**} \to X$, $\|P\| > 1$ with respect to the norm. After this, Godun gave in [5] a more general result that each nonreflexive Banach space $X$ admits an equivalent norm $\|\cdot\|$ such that for each projection $P$: $X^{**} \to X$, $\|P\| \geq 2$ with respect to the norm. These results are all related to the existence of an equivalent norm which admits no preduals. On the contrary, we are concerned with equivalent norms which admit preduals. In this paper we consider the class of such equivalent norms and demonstrate that given a nonreflexive Banach space $X$ "most" of the equivalent norms on $X$ do not admit preduals in the above sense.

Let $(\mathfrak{N}, |\cdot|)$ be a nonreflexive Banach space with norm $|\cdot|$. We denote by $\mathfrak{N}(X)$ the class of all the equivalent norms on $X$ and by $\mathfrak{N}_p(X)$ the class of all equivalent norms on $X$ which admit preduals. The purpose of this paper is to show in terms of metric space theory that $\mathfrak{N}_p(X)$ is a meager subset of the space $\mathfrak{N}(X)$. To accomplish this we introduce a metric $\rho$: $\mathfrak{N}(X) \times \mathfrak{N}(X) \to [0, \infty)$ on $\mathfrak{N}(X)$ defined by

$$\rho(|\cdot|_1, |\cdot|_2) = \log\{\inf\{\lambda > 0: |x|_1 \leq \lambda |x|_2 \leq \lambda^2 |x|_1 \quad \text{for all } x \in X\}\}.$$

Then $\rho$ defines a complete metric on $\mathfrak{N}(X)$, and it is shown (see Theorem 2 below) that $\mathfrak{N}_p(X)$ is nowhere dense in $\mathfrak{N}(X)$ with respect to the metric topology $\rho$. This means that $\mathfrak{N}_p(X)$ is meager in $\mathfrak{N}(X)$ and in this sense "most" of the equivalent norms do not admit preduals.