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Abstract homotopy theory and homotopy theory of functor category

Yoshimi SHITANDA

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Introduction

Many topologists have studied the various homotopy theories (the equivariant homotopy theory [2, 13] and the ex-homotopy theory [8, 21] etc.). Such homotopy theories have the common back-grounds. For example Puppe's theorem [19] and J. H. C. Whitehead's theorem [25] etc. hold also in these homotopy theories. Therefore these phenomena must be treated in the systematic manner. In this paper we study an axiomatization of homotopy theories mentioned above and deduce the various fundamental theorems systematically. Moreover we study the homotopy theory of the functor category in detail. Our theory enable us to treat the n-ad homotopy theory and equivariant homotopy theory in the unified manner. By introducing the cell structure in the functor category, we obtain the detailed results (e. g. the obstruction theory).

In part I we give an axiomatization of homotopy theory based on the cylinder and path functors (cf. [9, 10]). By introducing the extension condition and natural homotopy axioms 1, 2, we can obtain the various fundamental theorems (e. g. Puppe's theorem). Our axiomatization satisfies the duality principle and is closed under the constructions of the functor category and the comma category. Hence we can obtain the various fundamental theorems in the various homotopy theories in the systematic manner (cf. [8, 13, 16]).

In part II we study the homotopy theory of the functor category in detail. Let $\mathscr{D}$ be a topological small category, CGH the category of compactly generated Hausdorff spaces and continuous mappings. Let $\mathscr{F}$ = Cont Funct ($\mathscr{D}$, CGH) be the functor category whose objects are continuous contravariant functors and morphisms are natural transformations between them. The category $\mathscr{F}$ becomes an abstract homotopy category in the sense of Part I. By introducing $\mathscr{D}$-orbits D_a: $\mathscr{D} \to \mathrm{CGH}$ defined by $D_a(x) = \mathscr{D}(x, a)$ (hom-set in $\mathscr{D}$), we define the functor complex over $\mathscr{D}$ which is the natural generalization of equivariant CW complexes (cf. (2, 13)) and n-ad CW complexes. By the the natural isomorphism $\mathscr{F}(D_a \times T, X) = \mathrm{CGH}(T(a), X(a))$ where T is a constant functor and X a continuous functor, it is shown that Puppe's theorem, the celllar approximation theorem and J. H. C. Whitehead's theorem hold also in the category of functor complexes. We can also develop the obstruction theory