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Nonoscillatory solutions of neutral differential equations

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1. Introduction

In this paper we are concerned with neutral differential equations of the form

$$\frac{d^n}{dt^n}[x(t) - h(t)x(\tau(t))] + \sigma p(t)f(x(g(t))) = 0\,, \tag{1.1}$$

where $n \geq 2$, $\sigma = 1$ or -1, and the following conditions are always assumed to hold:

(1.2) $\tau(t) \in C[a, \infty)$, τ is nondecreasing on $[a, \infty)$, $\tau(t) < t$ for $t \geq a$ and $\lim_{t\to\infty} \tau(t) = \infty$;

(1.3) $h(t) \in C[\tau(a), \infty)$, $|h(t)| \leq h < 1$ for $t \geq a$, where h is a constant, and $h(t)h(\tau(t)) \geq 0$ for $t \geq a$;

(1.4) $p(t) \in C[a, \infty)$ and $p(t) > 0$ for $t \geq a$;

(1.5) $f(u) \in C((-\infty, \infty)\backslash\{0\})$ and $f(u)u > 0$ for $u \neq 0$;

(1.6) $g(t) \in C[a, \infty)$ and $\lim_{t\to\infty} g(t) = \infty$.

By a solution of (1.1) we mean a continuous function x which is defined and satisfies (1.1) on $[T_x, \infty)$ for some $T_x \geq a$ (so that $x(t) - h(t)x(\tau(t))$ is n-times continuously differentiable on $[T_x, \infty)$). Such a solution is said to be nonoscillatory if it has no zeros on $[T, \infty)$ for some $T \geq T_x$.

Recently there has been an increasing interest in the study of neutral differential equations, and a number of results have been obtained. For typical results we refer in particular to the papers [1–9, 14–18]. In this paper we make an attempt to study in a systematic way the structure of the set of nonoscillatory solutions of equation (1.1). In Section 2 we discuss the relation between two functions $x(t)$ and $x(t) - h(t)x(\tau(t))$. The results obtained in Section 2 will be effectively used in subsequent sections. In Section 3 we classify the nonoscillatory solutions of (1.1) into several classes according to the asymptotic behavior as $t \to \infty$. In Sections 4 and 5 we establish necessary and sufficient conditions for the existence of nonoscillatory solutions of (1.1) with specific asymptotic properties as $t \to \infty$.