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Chiral models and the Einstein-Maxwell field equations

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1. Introduction

The main objective in this paper is to provide a geometric picture of solutions of a $(1 + 1)$-dimensional reduction for the $(1 + 3)$-dimensional principal chiral model taking values in an arbitrary linear algebraic group.

Let G be a closed subgroup of the group scheme GL_N and assume that G is defined over $\boldsymbol{R}$. The equations of motion for the $SO(1, 2)$-invariant chiral model on flat Minkowski space can be written

$$d(t * d\sigma \cdot \sigma^{-1}) = 0 \tag{1.1}$$

for $\sigma \in G(\boldsymbol{C}[[t, z]])$. Here t, z are real variables, d is exterior differentiation, and $*$ is the Hodge operator with respect to the Lorentz metric $(dt)^2 - (dz)^2$.

Let λ be a real parameter. Let $\mathscr{A}$ denote an algebra $\{a = \Sigma_{n \in \boldsymbol{Z}} a_n \lambda^n \in \boldsymbol{C}[[t, z, \lambda, \lambda^{-1}]];\ \mathrm{ord}\, a_n \geq n\}$, where $\mathrm{ord}\, \varphi = \sup \{k \in \boldsymbol{Z};\ \varphi \in (\boldsymbol{C}[[t, z]]t + \boldsymbol{C}[[t, z]]z)^k\}$. Set $\mathscr{A}^{\pm} = \mathscr{A} \cap \boldsymbol{C}[[t, z, \lambda^{\pm 1}]]$, $\mathscr{P}_G = G(\mathscr{A}^+)$ and $\mathscr{N}_G = \{g \in G(\mathscr{A}^-);\ g(t, z, \infty) = 1\}$. Then $G(\mathscr{A}) = \mathscr{N}_G \mathscr{P}_G$ (Lemma 2.3 and K. Takasaki [6, (3.17)]). This decomposition is used for solving (1.1).

THEOREM 1.1. *There exist* $w \in \mathscr{N}_G$ *and* $p \in \mathscr{P}_G$ *such that* $w^{-1} p = \gamma(z + \lambda t^2/2 + 1/2\lambda)$ *for each* $\gamma \in G(\boldsymbol{C}[[z]])$. *Furthermore, if we set* $\sigma = p(t, z, 0)$, *then* σ *is a unique solution of* (1.1) *with* $\sigma(0, z) = \gamma(z)$.

We give a proof of the theorem in §2 and derive an explicit formula for the solution σ with $\sigma(0, z) \in G(\boldsymbol{C}[z])$. Also we consider a transformation group for solutions of (1.1). As an application, we show in §3 a variant of the Geroch conjecture [3], that is to say, a real form $\mathscr{SU}(1, 2)$ of $SL_3(\boldsymbol{C}[[z]])$ acts transitively on the space of plane wave solutions of the Einstein-Maxwell field equations.

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2. The chiral models

To start with, we consider a manifest invariance of (1.1). We note that $d(t * d\tau^{-1} \cdot \tau) = -\mathrm{Ad}\, \tau^{-1}(d(t * d\tau \cdot \tau^{-1}))$ for any $\tau \in G(\boldsymbol{C}[[t, z]])$. The following result is obvious.