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Discrete initial value problems and discrete parabolic potential theory

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§ 1. Introduction

In this paper, we shall study a discrete analogue of the initial value problems and the potential theory for the heat equation $\Delta u = \partial u/\partial t$, the potential theory established e.g. in Doob [1; 1. XV & XVII], Watson [4] and, in a more abstract form, in Maeda [3]. We choose an infinite network N and consider a "discrete cylinder" with base space N.

More precisely, let X be a countable infinite set of nodes, Y be a countable infinite set of arcs and K be the node-arc incidence function. We assume that the graph $\{X, Y, K\}$ is connected and locally finite and has no self-loop. Let r be a strictly positive real function on Y. We call the quartet $N = \{X, Y, K, r\}$ an infinite network (cf. [5], [6]). Next, let T be the set of all integers which will be regarded as the time space. For $s \in T$, put $T_s = \{t \in T; t \geq s\}$. We call $\{N, T\}$ (resp. $\{N, T_s\}$) the discrete cylinder (resp. discrete half-cylinder) with base N.

We set $\Xi = X \times T$ and denote by $L(\Xi)$ the set of all real functions on Ξ. For $u \in L(\Xi)$, we shall define the discrete (partial) derivatives du and ∂u and the Laplacian Δu. The operators d and Δ act on the variable $x \in X$ and ∂ on $t \in T$. The parabolic operator Π acting on $u \in L(\Xi)$ is defined by

$$\Pi u(\xi) = \Delta u(\xi) - \partial u(\xi), \ \xi = (x, t) \in \Xi.$$

Our initial value problems and potential theory will be discussed with respect to this operator Π.

For our study, we first recall in § 2 some properties of the 1-Green function of N relative to the equation $\Delta u = u$, and give some results on iterations of the 1-Green operators. In § 3, we consider superparabolic functions on a set in Ξ and give minimum principles. We study in § 4 an initial value problem on $\{N, T_s\}$. The existence and uniqueness of the parabolic Green function G_α of $\{N, T\}$ with pole at $\alpha \in \Xi$ will be studied in § 5. Solutions of an initial boundary value problem as well as the parabolic Green function of $\{N, T\}$ will be constructed by means of the iterations of the 1-Green operator of N. In case N has the harmonic Green function g_a with pole at $a \in X$, we have the