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Infinite families of non-principal prime ideals

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Abstract. Bouvier [2] has shown that if A is a Krull domain, then the set of non-principal prime ideals of height one of A is either empty or infinite. Here we prove a similar result under less restrictive hypotheses.

Let A be a domain and let $\mathscr{P}$ be a family of prime ideals of A, satisfying the following conditions:

1) $\bigcap_{n=0}^{\infty} A_P P^n = 0$ for every $P \in \mathscr{P}$.

2) $\bigcap_{P \in \mathscr{P}} A_P = A$.

3) If $f \in A$, $f \neq 0$, there exist only finitely many ideals $P \in \mathscr{P}$ such that $f \in P$.

4) If $n > 1$ and $P_1, \ldots, P_n$ are distinct prime ideals in the family $\mathscr{P}$, there exists f such that $f \in P_1 \backslash P_1^2$, $f \notin P_2 \cup \cdots \cup P_n$.

For example, we may take $\mathscr{P}$ to be the family of prime ideals of height one of a Krull domain, as it was done by Bouvier.

Following suggestions of W. Heinzer, we indicate other examples of domains and families of prime ideals satisfying conditions (1)–(4).

The family of all prime ideals of height one of a noetherian domain, in which every principal ideal has no embedded primes, satisfies the conditions (1)–(4). These are precisely the prime ideals of height one of domains satisfying the S-sub-2-condition of Serre, and include the Cohen-Macaulay domains.

In a still unpublished paper, Barucci, Gabelli & Roitman [1] study the semi-Krull domains, introduced earlier by Matlis [3]: the family of prime ideals of height one satisfies also conditions (1)–(4).

We note that condition (4) implies:

4′) If $P, P' \in \mathscr{P}$, $P \neq P'$, then P, P' are incomparable by inclusion.

LEMMA. *If P is a prime ideal of A satisfying condition* (1) *and $A_P P$ is a principal ideal, then A_P is the ring of a discrete valuation of height one of the field of quotients of A.*

PROOF. If $x \in A_P$, $x \neq 0$, let $v_P(x) = n$ be the unique integer such that $x \in A_P P^n \backslash A_P P^{n+1}$; let also $v_P(0) = \infty$. If $x, y \in A_P$ it is obvious that $v_P(x + y) \geq$