HIROSHIMA MATH. J.
22 (1992), 33–56

# Periodic zeta functions for rank 1 space forms of symmetric spaces

Leticia BARCHINI[1] and Floyd L. WILLIAMS[2]

(Received September 20, 1990)

## 1. Introduction

For the modular group $\Gamma = PSL(2, Z)$ and a positive number $\alpha$, $A$. Fujii [5], [6] has studied a periodic zeta function

$$(1.1) \qquad Z_\alpha(s) = \sum_{r_j > 0} \frac{\sin \alpha r_j}{r_j^s} \qquad \mathrm{Re}\, s > 1$$

associated with the discrete spectrum $0 = \lambda_0 \leq \lambda_1 \leq \cdots$ of the Laplace-Beltrami operator acting on $L^2(\Pi^+/\Gamma)$ where $\Pi^+$ is the upper half-plane. Here, as usual, $r_j$ is given by $\lambda_j = \frac{1}{4} + r_j^2$. Using the Selberg trace formula Fujii proves that $Z_\alpha$ has an analytic continuation $Z_\alpha$ to the whole plane—ie. $Z_\alpha$ is an entire function. Among other results he also proves that

$$(1.2) \qquad \lim_{\alpha \to \log N(P_1)} (\alpha - \log N(P_1)) Z_\alpha(0) = \frac{1}{2\pi} \sum_{\{P\},\, N(P) = N(P_1)} \tilde{\Lambda}(P)/\sqrt{N(P)}$$

where $\{P_1\}$ is any hyperbolic conjugacy class, $N$ is the norm function and $\tilde{\Lambda}$ is the von Mangoldt function for the Selberg zeta function. Some related work appears in [2], [4], [10], [14].

It seems natural to replace $\Pi^+$ by a general rank one symmetric space $G/K$ where $G$ is a connected non-compact semisimple Lie group with finite center and $K$ is a maximal compact subgroup of $G$. A suitable version of the trace formula is available in this context for $\Gamma$ a discrete subgroup of $G$. In this paper we consider indeed a corresponding zeta function $Z_\alpha$, as in (1.1), and prove that $Z_\alpha$ extends to an entire function on the complex plane at least when $G$ is simple and $\Gamma$ is without torsion and is co-compact. Actually we construct an infinite family $\{Z_{\alpha,b}\}_{b \geq 0}$ of zeta function with $Z_{\alpha,0} = Z_\alpha$. Each $Z_{\alpha,b}$ is entire; see Theorems 5.17 and 6.10.

For the modular group $\Gamma$ one has the well known fact that $\lambda_1 > \frac{1}{4}$; ie. no complementary series representations of $PSL(2, R)$ occur in the discrete spectrum of $L^2(\Gamma | PSL(2, R))$. However, in the case at hand complementary

[1] Research supported by CONICET, Argentina
[2] Research supported by NSF Grant No. DMS-8802597