HIROSHIMA MATH. J.
22 (1992), 23–31

Topology of moduli space of certain $SU(2)$ connections of degree 2 over S^4

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(Received September 18, 1990)

1. Introduction

Let $SU(2) \to P_k \to S^4$ be the principal $SU(2)$ bundle of degree $c_2(P_k) = k$ and let $\mathscr{A}_k$ (resp. $\mathscr{B}_k$) be the set of anti-self-dual connections (resp. $SU(2)$ connections) over P_k. The restricted gauge group (consisting automorphisms which are the identity on the base point $\infty \in S^4$) acts on $\mathscr{A}_k$ and $\mathscr{B}_k$. We define M_k and $\mathscr{M}_k$ to be the orbit space of $\mathscr{A}_k$ and $\mathscr{B}_k$ by the restricted gauge group respectively. M_k is called the framed moduli space of instantons of degree k and $\mathscr{M}_k$ the framed moduli space of $SU(2)$ connections of degree k.

M_k is described by the linear algebra known as the ADHM (Atiyah-Drinfeld-Hitchin-Manin) construction [2]. In the ADHM construction, the following three conditions are imposed: (1) symmetric condition, (2) rank condition and (3) reality condition. The space obtained by imposing the conditions (1) and (2) only is denoted by $\hat{M}_k$ and we have inclusions i_1 and i_2:

$$M_k \xrightarrow{i_1} \hat{M}_k \xrightarrow{i_2} \mathscr{M}_k.$$

We see easily that $M_1 = \hat{M}_1$. So, we shall compare the topology of M_2 with that of $\hat{M}_2$.

THEOREM A. *$\hat{M}_2$ is connected, $\pi_1(\hat{M}_2) = \mathbf{Z}_2$ and $\pi_2(\hat{M}_2) = \mathbf{Z}$. Moreover, $i_{1*}: H_*(M_2; \mathbf{Z}_2) \to H_*(\hat{M}_2; \mathbf{Z}_2)$ is an isomorphism.*

It is known that M_1 is diffeomorphic to $SO(3) \times \mathbf{R}^5$ [2]. The topology of M_2 is studied in [6] and the structures of $H_*(M_2; \mathbf{Z}_2)$ and $H^*(M_2; \mathbf{Z}_2)$ are completely determined in [8]. It is known also that $\pi_1(M_k) = \mathbf{Z}_2$ for all k [7]. Appropriate modifications of Hurtubise's proof might show that $\pi_1(\hat{M}_k) = \mathbf{Z}_2$ hold for all k.

This paper is organized as follows. In §2 we shall review the ADHM construction and give the precise definition of $\hat{M}_k$ as $\hat{F}_k/O(k)$. In §3 we shall construct non-trivial elements in $H_*(\hat{M}_2; \mathbf{Z}_2)$ by using the methods of Boyer and Mann [4] and as an application, we shall estimate the $\mathbf{Z}_2$ coefficient Betti numbers of $\hat{M}_2$ from below. In §4 we shall prove a proposition which determines $H_*(\hat{F}_2; \mathbf{Z}_2)$. In §5 and 6 we shall prove Theorem A by using the