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Ascending chain conditions on special classes of ideals of Lie algebras

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0. Introduction

The class Max-$\triangleleft$ of Lie algebras with ascending chain condition on ideals, otherwise known as noetherian Lie algebras, has been studied by several authors, see Aldosray [1], Amayo and Stewart [3, 4], Kawamoto [10], Kubo and Honda [12], and Stewart [14, 15]. Less stringent chain conditions can be imposed by requiring the ascending chain condition only on certain special classes of ideals. For example Aldosray and Stewart [2] study the class Max-c of Lie algebras with ascending chain condition on centralizer ideals; and Ikeda [6], Kubo [11], and Tôgô [17] study ascending chain conditions on generalized soluble ideals.

Here we consider the classes Max-ci, Max-ess, and Max-smi of Lie algebras with the ascending chain condition on complement ideals, essential ideals, and small ideals, defined respectively in §§1, 2, 5 below. Our aim is to elucidate the basic properties of these classes and the relations between them.

In §1 we study the ascending chain condition on complement ideals and give a number of examples to show that most standard properties of Max-$\triangleleft$ fail for such algebras. In §2 we introduce the 'dual' concepts of essential ideals and small ideals, and the singular ideal. The main result, Theorem 2.8, shows that the Lie product of a pair of essential ideals is always essential if and only if the Lie algebra is semisimple. §3 is devoted to questions of the following kind. Suppose that every quotient L/I of L by nonzero ideals I (possibly of some special type) satisfies some ascending chain condition. Does L also satisfy this chain condition? We show that the answer is affirmative for Max-$\triangleleft$ (I any ideal); Max-ci (I a complement ideal); and Max-smi (I a small ideal). It is *negative* for Max-c (even if I may be any nonzero ideal), answering Question 3 of Aldosray and Stewart [2]. However, it is affirmative for Max-c when L is semisimple (I any centralizer ideal).

Camillo [5] proves that a commutative ring in which every quotient is Goldie must be noetherian. In §4 we show that the natural Lie algebra analogue of Camillo's theorem is *false.* However, using results of Shock [13]