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Congruences between binomial coefficients $\binom{2f}{f}$ and Fourier coefficients of certain η-products

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§ 1. Introduction

Let k and l be positive integers with $(k, l) = 1$. Let p be a prime, $p \equiv l$ *mod* k and the integer f is defined by $p = kf + l$. We consider the congruences modulo p of binomial coefficients of the form $\binom{2f}{f}$. In the classical results, for $k = 4$ and $l = 1$, Gauss proved that

$$\binom{2f}{f} \equiv 2a \ mod\ p,$$

where $p = a^2 + b^2 = 4f + 1$ and $a \equiv 1$ *mod* 4. For $k = 3$ and $l = 1$, Jacobi proved that

$$\binom{2f}{f} \equiv -a \ mod\ p,$$

where $4p = a^2 + 27b^2$ and $a \equiv 1$ *mod* 3. Moreover, the number $2a$ (resp. $-a$) can be regarded as the p-th Fourier coefficient of the cusp form of CM-type associated with the Hecke character of $\mathbb{Q}(\sqrt{-1})$ (resp. $\mathbb{Q}(\sqrt{-3})$). In the recent results, for $l = 1$ and $k \leq 24$, these were studied by Hudson and Williams [4] using Jacobi sums.

In this paper, we shall prove the congruence properties between binomial coefficients $\binom{2f}{f}$ and Fourier coefficients of certain η-products:

THEOREM 1. *Let k and l be the above and put $m = 4l/k$. Write*

$$\sum_{n=1}^{\infty} \gamma_n^{(k,l)} q^n = \eta(k\tau)^2 \eta(2k\tau)^{1+m} \eta(4k\tau)^{3-3m} \eta(8k\tau)^{2m-2}.$$

where $\eta(\tau) = q^{1/24} \prod_{n=0}^{\infty} (1 - q^n)$ is the Dedekind η-function with $q = e^{2\pi i\tau}$ and