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Imaginary powers of elliptic second order differential operators in L^p-spaces

Dedicated to Professor Y. Komura on the occasion of his 60th-birthday

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1. Introduction

Let Ω be an open domain in $\mathbb{R}^n$ with $n \geq 1$ and consider the second order elliptic differential operator on Ω defined by

$$(Eu)(x) = -\sum_{j,k=1}^{n} a_{jk}(x)\partial_j\partial_k u(x) + \sum_{j=1}^{n} b_j(x)\partial_j u(x) + c(x)u(x) \qquad (1.1)$$

where $\partial\Omega$ is the boundary of Ω. Let $1 < p < \infty$ and let E_p denote the $L^p(\Omega)$-realization of this boundary value problem defined by $(E_p u)(x) = (Eu)(x)$ with domain of definition $\mathscr{D}(E_p) = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega)$. The aim of this paper is to give simple conditions for the boundedness of the purely imaginary powers E_p^{iy} and for an estimate of the form

$$\| E_p^{iy} \| \leq K e^{\theta|y|} \qquad \text{for all } \ y \in \mathbb{R}, \qquad (1.2)$$

where $K = K(\theta) > 0$, $0 < \theta < \pi$ are constants; see Section 3 for the definition of complex powers of linear operators. We show that this holds, under some smoothness assumptions and technical restrictions on the coefficients a_{jk}, b_j, c and the domain Ω, essentially whenever the resolvent estimate

$$\|(\lambda + E_p)^{-1}\| \leq \frac{C}{|\lambda|} \qquad \text{for all } \ \lambda \in \Sigma_{\pi-\theta} = \{\lambda \in \mathbb{C}: \lambda \neq 0, |\arg \lambda| < \pi - \theta\}, \quad (1.3)$$

is satisfied. Observe that this condition for $0 < \theta < \pi/2$ implies that $-E_p$ generates an analytic semigroup e^{-tE_p} in $L^p(\Omega)$. In Prüss and Sohr [16] it has been shown that (1.3) is always necessary for (1.2); the difficult part is the proof of the converse. In the paper just mentioned it is also shown that (1.2) implies the same estimate for $\delta + E_p$ for each $\delta > 0$, with the same constants K and θ.

Estimates for the imaginary powers A^{iy} of a closed linear operator A in a Banach space X are of interest for several reasons. One of these is related to the determination of the domains $\mathscr{D}(A^\alpha)$ of the fractional powers A^α of A,