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Closed ideals of Lie algebras

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0. Introduction

In 1960 Goldie [7] showed how to develop a structure theory for semiprime rings with maximum condition in terms of what he called *closed ideals*. An alternative and slightly different definition was given by Lesieur and Croisot [9] and used by Divinsky [6]. The aim of this paper is to define analogous concepts for Lie algebras, and to establish their basic properties.

In § 1 we introduce two analogous notions for Lie algebras, which we call *closed ideals* and **closed ideals* to distinguish them. They are defined in terms of the *closure* $cl(I)$ and **closure* $cl^*(I)$ of an ideal I, see Definitions 1.1 and 1.2. We show that $cl(I) \subseteq cl^*(I)$ and that the two closures coincide for semisimple Lie algebras (defined below). The basic properties of closed ideals are established in § 2, where we show in particular that the closure of an ideal need not be an ideal—indeed it need not even be a vector subspace. Analogous questions for cl^* are investigated in § 3; in contrast, the *closure of an ideal is always an ideal.

In § 4 we study semisimple algebras. The main result is that the following four concepts are equivalent for semisimple algebras: centralizer ideal, complement ideal, closed ideal, and *closed ideal. In § 5 we discuss, for arbitrary Lie algebras, relations between centralizer ideals, complement ideals, closed ideals, *closed ideals, and ideals with no proper essential extension, where the latter concept is analogous to one defined for rings in Behrens [5] and Goodearl [8]. The main result is that *closed ideals are always closed; closed ideals are always complement ideals; and being a complement ideal is equivalent to having no proper essential extension. Moreover, no other implications between these properties are valid. We also show that the sum of two complement (respectively closed) ideals need not be a complement (respectively closed) ideal. Finally, in § 6, we investigate various chain conditions on closed and *closed ideals, extending work in Aldosray and Stewart [3] and answering part of Question 1.7 of that paper. In particular we show that the ascending chain condition for complement ideals is equivalent to the descending chain condition for complement ideals.

All Lie algebras considered are of finite or infinite dimension over a field **k** of arbitrary characteristic, unless otherwise specified. Most notation used