HIROSHIMA MATH. J.
24 (1994), 607–612

On the upper bounds of Green potentials

Dedicated to Professor M. Nakai on the occasion of his 60th birthday

Hiroaki AIKAWA

(Received June 14, 1993)

1. Introduction

Let D be a domain in $\boldsymbol{R}^n$ $(n \geq 2)$ with the Green function $G(x, y)$ for the Laplace equation. By $|D|$ we denote the volume of D. In [5] Cranston and McConnell proved the following result.

THEOREM A. *Let $n = 2$ and let D be a domain of finite area. Then there exists an absolute constant c such that for any function $h \geq 0$ harmonic on D,*

$$\int_D G(x, y)h(y)dy \leq c|D|h(x).$$

Their methods are highly probabilistic; they use the life time of conditioned Brownian motion. Chung [4] gave a simplified proof of Theorem A. His proof is based on the up-crossing and the down-crossing inequalities in the martingale theory. Bañuelos [2] extended Theorem A to general elliptic equations and $n \geq 3$. (For the higher dimensional case we need to assume some boundary regularity.) His proof is also probabilistic.

The purpose of this note is to give an elementary analytic proof of Theorem A. Throughout this note we let h be a positive harmonic function on D. We say that u is an h-Green potential of density f if

$$u(x) = \frac{1}{h(x)} \int_D G(x, y)f(y)dy.$$

In other words, u is the h-Green potential of density f if hu is the Green potential of density f. In this terminology, the conclusion of Theorem A reads as follows: *the upper bound of the h-Green potential of density h is dominated by $c|D|$.*

Let us consider the upper bound of h-Green potentials. In the simplest

Part of this work was made while the author was visiting the Department of Mathematics, University of Uppsala. He wishes to thank the Royal Swedish Academy of Sciences and the Japan Society of Promotion of Science.