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The max-MSE's of minimax estimators of variance in nonparametric regression

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1. Introduction and notations

Consider the nonparametric regression model

$$Y_i = g(t_i) + \varepsilon_i, \qquad 1 \le i \le n,$$

where observations are taken at design points t_i for $1 \le i \le n$, and the errors ε_i are independent and identically distributed as normal distribution with mean zero and variance σ^2. The response function g is assumed to belong to the space $W = \{g : g$ and g' are absolutely continuous, and $\int_0^1 |g''(t)|^2 dt < \infty\}$.

We deal with minimax estimators of σ^2 defined in Buckley, Eagleson and Silverman [1]. They are based on a restricted class of the response functions $W_C = \{g \in W : \int_0^1 |g''(t)^2 dt \le C\}$. Define the max-MSE criterion as

$$M(\hat{\sigma}^2; \sigma^2, C) = \max_{g \in W_C} \frac{1}{\sigma^4} E(\hat{\sigma}^2 - \sigma^2)^2$$

for any given estimator $\hat{\sigma}^2$ of σ^2. To simplify the minimax problem, we shall use a natural coordinate system. Demmler and Reinsch [2] showed that there is a basis for the natural cubic splines, $\phi_1(\cdot), \ldots, \phi_n(\cdot)$, determined essentially uniquely by

$$\sum_{i=1}^n \phi_j(t_i)\phi_k(t_i) = \delta_{jk}, \qquad \int_0^1 \phi_j''(t)\phi_k''(t)dt = \delta_{jk}\omega_k$$

with $0 = \omega_1 = \omega_2 < \cdots < \omega_n$. Here $\delta_{jk} = 1$ if $j = k$ and 0 otherwise. Let $\tilde{y} = (Y_1, \ldots, Y_n)^T$ and $\tilde{g} = (g(t_1), \ldots, g(t_n))^T$ be the vectors expressed with respect to a natural basis of R^n, $\{(\phi_j(t_i))\}$. Our attention is restricted to a class of estimators of σ^2 whose form is $\hat{\sigma}^2(D) = \tilde{y}^T D \tilde{y}/\mathrm{tr}\, D$, $D \in \Delta$. Here Δ is the class of $n \times n$ symmetric non-negative definite matrices D for which $\hat{\sigma}^2(D)$ is unbiased when g is a straight line. Buckley, Eagleson and Silverman [1] proposed minimax estimators defined as the estimator which minimizes $M(\hat{\sigma}^2(D); \sigma^2, C)$ over $D \in \Delta$. Their minimax estimators depend on σ^2 and C through C/σ^2. The explicit expressions of them were obtained in Fujioka [3] as follows. Putting $\omega_i^+(r) = \omega_i(1 + 4\omega_i/r)^{-1/2}$ for $3 \le i \le n$, we set for $3 \le k \le n-1$