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On absolutely continuous invariant measures with respect to Hausdorff measures on self-similar sets

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ABSTRACT. We treat here measures which are invariant with respect to the renormalization map on a self-similar set. A criterion for their absolute continuity with respect to their associated Hausdorff measures is given in terms of symbolic dynamics. Using this criterion, we give a striking characterization of the equilibrium state for a certain potential function.

1. Introduction

In this paper we shall study the properties of ergodic invariant measures on self-similar sets. Especially we shall investigate the relations between invariant measures and the Hausdorff measures associated to them from the point of the absolute continuity. In the sense of [8], a self-similar set K is constructed from a system $\Phi = \{\varphi_i \mid i \in S\}$, $S = \{1, \cdots, N\}$ of contractions and a bounded open set $V \subset \mathbf{R}^d$ by $K = \bigcap_{n=1}^{\infty} \bigcup_{(i_1, \dots, i_n) \in S} \varphi_{i_1} \circ \cdots \circ \varphi_{i_n}(\bar{V})$. Here we assume that Φ and V satisfy $\bigcup_{i=1}^{N} \varphi_i(V) \subset V$ and $\varphi_i(V) \cap \varphi_j(V) = \emptyset$ if $i \neq j$, which is often referred to as the open set condition. See [8] for the details. It is easy to see that there exists a continuous surjection $\psi: S^{\mathbf{N}} \to K$ and this fact enables us to work on the symbolic dynamics $(S^{\mathbf{N}}, \sigma)$, $\sigma((x_n)_{n \in \mathbf{N}}) = (x_{n+1})_{n \in \mathbf{N}}$. In order to guarantee the existence of the renormalization transformation f on K corresponding to σ with a measure theoretically negligible exceptional set, we have to make a further assumption (A7) below. By virtue of this assumption we can immediately have a one to one correspondence between f-invariant measures and σ-invariant measures through the map ψ. In this paper we shall make a little extension of the above notion of self-similar sets. That is, as in [1], for each $N \times N$ aperiodic matrix A we construct a compact set K_A by $K_A = \bigcap_{n=1}^{\infty} \bigcup_{(i_1, \dots, i_n) \in \Sigma_{A,n}} \varphi_{i_1} \circ \cdots \circ \varphi_{i_n}(\bar{V})$ where $\Sigma_{A,n}$ is the set of all A-admissible words of length n, that is, those $(i_1, \cdots, i_n) \in S^n$ such that $A_{i_j i_{j+1}} = 1$, $j = 1, \cdots, n-1$. Deterministic Markov self-similar sets which are considered

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