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Limit sets and square roots of homeomorphisms

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ABSTRACT. In this paper, we define the positive and negative limit sets and characterize their dynamical properties in terms of the non-Hausdorff sets. By using these sets, we also consider the condition that a given homeomorphism has a square root, and give an example of a wandering homeomorphism without square roots.

1. Introduction

For a given homeomorphism f, a homeomorphism g satisfying $g \cdot g = f$ is called a *square root* of f. Although the square root is not unique in general, we always denote g by $\sqrt{f}$. In this paper, we consider the condition that a given homeomorphism of a non-compact space has a square root. For this purpose, we use the positive and negative limit sets, which will be defined and characterized in §2.

For homeomorphisms of compact spaces, we can use the nonwandering sets to show the non-existence of square roots as follows: Let f be a homeomorphism. For any homeomorphism h, the nonwandering set $\Omega(f)$ of f and the nonwandering set $\Omega(hfh^{-1})$ of hfh^{-1} satisfy the relation $h\Omega(f) = \Omega(hfh^{-1})$. Since the square root $\sqrt{f}$ commutes with f, $\Omega(f)$ is also invariant under $\sqrt{f}$. By using this fact, we can construct a homeomorphism which has no square roots: We choose a homeomorphism f of S^2 such that f preserves each curve of Figure 1 and exchanges two thorns on the equator. In particular, the equator with two thorns is invariant under f, and f exchanges their branch points. If f has a square root $\sqrt{f}$, then the equator with two thorns is also invariant under $\sqrt{f}$ because $\Omega(f)$ consists of two fixed points and the equator with two thorns. Thus $\sqrt{f}$ either exchanges or fixes the branch points. However this contradicts the assumption that its square f exchanges them.

The above argument cannot be applied to study the square roots of homeomorphisms of non-compact spaces with empty nonwandering set. Thus

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