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# Hypersingular integrals and Riesz potential spaces

*Dedicated to Professor Fumi-Yuki Maeda on the occasion of his sixtieth birthday*

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ABSTRACT. We introduce Riesz potential spaces and give the characterization in terms of hypersingular integrals.

## 1. Introduction and preliminaries

For a function $u(x)$ on the $n$-dimensional Euclidean space $R^n$ $(n \geq 3)$, the difference $\Delta_t^\ell u(x)$ and the remainder $R_t^\ell u(x)$ of order $\ell$ with increment $t = (t_1, \ldots, t_n) \in R^n$ are defined by

$$\Delta_t^\ell u(x) = \sum_{j=0}^{\ell} (-1)^j \binom{\ell}{j} u(x + (\ell - j)t),$$

$$R_t^\ell u(x) = u(x+t) - \sum_{|\gamma| \leq \ell - 1} \frac{D^\gamma u(x)}{\gamma!} t^\gamma$$

where $\gamma$ is a multi-index $(\gamma_1, \ldots, \gamma_n)$, $t^\gamma = t_1^{\gamma_1} \cdots t_n^{\gamma_n}$, $D^\gamma = D_1^{\gamma_1} \cdots D_n^{\gamma_n}$ $(D_j = \partial/\partial x_j)$, $\gamma! = \gamma_1! \cdots \gamma_n!$ and $|\gamma| = \gamma_1 + \cdots + \gamma_n$. Since $R_t^\ell u(x)$ is the remainder of Taylor's formula, we obviously see that

(1.1) $\quad R_t^\ell u(x) = 0$ for all $t \in R^n \Leftrightarrow u$ is a polynomial of degree $\ell - 1$

for $C^\infty$-functions $u$. We also have ([6: p. 1102])

(1.2) $\quad \Delta_t^\ell u(x) = 0$ for all $t \in R^n \Leftrightarrow u$ is a polynomial of degree $\ell - 1$

for locally integrable functions $u$. Using the difference and the remainder, for $\alpha > 0$ and a positive integer $\ell$, we define the singular difference integral $D^{\alpha,\ell} u$

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