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A chiral model related to the Einstein equation

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ABSTRACT. We construct some new rational solutions of the stationary axisymmetric Einstein equation.

0. Introduction

Our main objective in this paper is to construct a family of solutions of a field equation for $\sigma \in \mathrm{gl}(2, C[[t^{-1}, t, z]])$:

$$d*(td\sigma \cdot \sigma^{-1}) = 0, \tag{0.0}$$

where $*$ denotes the Hodge operator with a Lorentz metric $(dt)^2 - (dz)^2$ (i.e. $*dt = dz$, $*dz = dt$). This chiral model is the main part of the Einstein equation for a cyclindrical wave ansatz. Moreover, the equation of motion for the Ernst potential is written in a matrix form above. So the chiral model (0.0) is important in construction of exact vacuum gravitational fields, and much progress has been made on the inverse scattering method and universal Grassmann manifold approach [2], [3], [4], [5], [6].

Here, we seek solutions of (0.0) by a dressing method. Taking account of $d*(td \log t^s) = 0$, we consider an ansatz $\sigma = \tau \cdot \deg(t^{s_1}, t^{s_2})$ with $\tau \in GL(2, C[[t, z]])$ and $s_1, s_2 \in Z$. If σ satisfies (0.0) and c is a constant matrix, then $\sigma \cdot t^s$ and $c^{-1} \cdot \sigma \cdot c$ also satisfy (0.0). Hence we may assume that $s_1 \geq 0$ and $s_2 = 0$, without loss of generality. We are mainly concerned with this ansatz, and we investigate its solutions in a group-theoretic viewpoint.

Let $A = C[[t, z]]$. For $a \in A$, we set $\operatorname{ord} a = \sup\{k \in Z; a \in (At + Az)^k\}$. Let $\mathscr{A}$ denote an algebra $\{a = \sum a_n \lambda^n \in A[[\lambda, \lambda^{-1}]]; \operatorname{ord} a_n + n \geq 0\}$. If $\psi = \sum \psi_n \lambda^n \in \mathrm{gl}(2, \mathscr{A})$ and $\psi_0 \in GL(2, A)$, then ψ has a unique decomposition $\psi = \psi^- \cdot \psi^+$ with $\psi^- = 1 + \sum_{k<0} \psi_k^- \lambda^k$ and $\psi^+ = \sum_{k \geq 0} \psi_k^+ \lambda^k$ ([10]). We refer to this as the Birkhoff decomposition. Then we can construct a solution of (0.0) as follows.

THEOREM 0.0. *Let $s \in Z_+$, $\phi \in GL(2, C[[x]])$ and assume that ϕ_{12}, $\phi_{21} \in$*

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