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Quantum deformations of certain prehomogeneous vector spaces I

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ABSTRACT. We shall construct a quantum analogue of the prehomogeneous vector space associated to a parabolic subgroup with commutative unipotent radical.

0. Introduction

Let $\mathfrak{g}$ be a simple Lie algebra over the complex number field $\mathbf{C}$, and let $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{m}^+$ be a parabolic subalgebra of $\mathfrak{g}$, where $\mathfrak{l}$ is a maximal reductive subalgebra of $\mathfrak{p}$ and $\mathfrak{m}^+$ is the nilpotent part. We denote by $\mathfrak{m}^-$ the nilpotent subalgebra of $\mathfrak{g}$ such that $\mathfrak{l} \oplus \mathfrak{m}^-$ is a parabolic subalgebra of $\mathfrak{g}$ opposite to $\mathfrak{p}$. Take an algebraic group L with Lie algebra $\mathfrak{l}$.

In this paper we shall deal with the case where $\mathfrak{m}^\pm$ is nonzero and commutative. Then $\mathfrak{m}^+$ consists of finitely many L-orbits.

Our aim is to give a quantum analogue of the prehomogeneous vector space $(L, \mathfrak{m}^+)$. More precisely, we shall construct a quantum analogue A_q of the ring $A = \mathbf{C}[\mathfrak{m}^+]$ of polynomial functions on $\mathfrak{m}^+$ as a noncommutative $\mathbf{C}(q)$-algebra endowed with the action of the quantized enveloping algebra $U_q(\mathfrak{l})$ of $\mathfrak{l}$, and show that for each L-orbit C on $\mathfrak{m}^+$ there exists a two-sided ideal $J_{C,q}$ of A_q which can be regarded as a quantum analogue of the defining ideal J_C of the closure $\bar{C}$ of C. Such an object was intensively studied in the cases $\mathfrak{g} = \mathfrak{sl}_n$ (see Hashimoto-Hayashi [3], Noumi-Yamada-Mimachi [10]) and $\mathfrak{g} = \mathfrak{so}_{2n}$ (see Strickland [13]).

Our method is as follows. Since $\mathfrak{m}^-$ is identified with the dual space of $\mathfrak{m}^+$ via the Killing form, A is isomorphic to the symmetric algebra $S(\mathfrak{m}^-)$. By the commutativity of $\mathfrak{m}^-$ the enveloping algebra $U(\mathfrak{m}^-)$ is naturally identified with the symmetric algebra $S(\mathfrak{m}^-)$. Hence we have an identification $A = U(\mathfrak{m}^-)$. Then using the Poincaré-Birkhoff-Witt type basis of the quantized enveloping algebra $U_q(\mathfrak{g})$ (Lusztig [9]) we obtain a natural quantization A_q of A as a subalgebra of $U_q(\mathfrak{g})$. The algebra A_q has a canonical generator system satisfying quadratic fundamental relations. In particular, it is a graded algebra. The adjoint action of $U_q(\mathfrak{g})$ on $U_q(\mathfrak{g})$ is defined using the Hopf

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