HIROSHIMA MATH. J.
29 (1999), 193–225

Contravariant forms on generalized Verma modules and *b*-functions

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(Received February 23, 1998)
(Revised May 14, 1998)

ABSTRACT. Two bilinear forms on a scalar generalized Verma module $M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} \mathbf{C}_\lambda$ are treated in this paper, where $\mathfrak{g}$ is a complex simple Lie algebra and $\mathfrak{p}$ is its parabolic subalgebra. They coincide on each $\mathfrak{l}$-irreducible component up to scalar multiple, where $\mathfrak{l}$ is a Levi subalgebra of $\mathfrak{p}$. These ratios have played important roles in the representation theory. We show intrinsically that these ratios are products of b-functions when the nilpotent radical $\mathfrak{n}^+$ of $\mathfrak{p}$ is commutative. As an application we explain the reason why the b-functions control the irreducibility or $M(\lambda)$, the orbit decomposition of $\mathfrak{n}^+$ under the action of the Levi subgroup, and the unitarizability of $M(\lambda)$.

1. Introduction

Let G be a complex simple Lie group. Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{h}$ its Cartan subalgebra. Let Δ and Δ^+ be the root system and the positive system, respectively. Let $\mathfrak{p}$ be a parabolic subalgebra containing $\mathfrak{h}$ and all the positive root spaces. Then the pair $(\mathfrak{g}, \mathfrak{p})$ is said to be of commutative parabolic type if the nilpotent radical $\mathfrak{n}^+$ of $\mathfrak{p}$ is commutative. In this paper, we exclusively consider $(\mathfrak{g}, \mathfrak{p})$ of commutative parabolic type.

Let $M(\lambda)$ be the scalar generalized Verma module induced from $\lambda \in \mathrm{Hom}(\mathfrak{p}, \mathbf{C})$. Then $M(\lambda) \simeq \mathbf{C}[\mathfrak{n}^+]$ as vector spaces. We therefore obtain the representation of $U(\mathfrak{g})$ on $\mathbf{C}[\mathfrak{n}^+]$, and denote it by $\Psi_\lambda : U(\mathfrak{g}) \to \mathrm{End}\, \mathbf{C}[\mathfrak{n}^+]$.

Let $\{X_\alpha, H_i\}$ be a Chevalley basis of $\mathfrak{g}$, where $X_\alpha \in \mathfrak{g}^\alpha$ for $\alpha \in \Delta$ and $H_i \in \mathfrak{h}$. To give the definition of contravariant forms, we define an involutive anti-automorphism ${}^t\cdot$ on $U(\mathfrak{g})$ by $X_\alpha \mapsto X_{-\alpha} (\alpha \in \Delta)$ and to be the identity on $\mathfrak{h}$. For a representation (π, V) of $\mathfrak{g}$, a bilinear form $(\ ,\ )$ on V is called a contravariant form or a $\pi(U(\mathfrak{g}))$-contravariant form if it satisfies $(\pi(X)v, w) = (v, \pi({}^tX)w)$ for $X \in \mathfrak{g}$ and $v,\ w \in V$. We study a canonical $\Psi_\lambda(U(\mathfrak{g}))$-contravariant form $(\ ,\ )_\lambda$ and a canonical $\mathrm{ad}(U(\mathfrak{l}))$-contravariant form $(\ ,\ )$ on $M(\lambda) \simeq \mathbf{C}[\mathfrak{n}^+]$, where $\mathfrak{l}$ is the Levi subalgebra of $\mathfrak{p}$ containing $\mathfrak{h}$. Let $\mathbf{C}[\mathfrak{n}^+] = \bigoplus_\mu I_\mu$ be the irreducible decomposition as an $\mathrm{ad}(U(\mathfrak{l}))$-

1991 *Mathematical Subject Classification*: 17B10.

Key words and Phrases: generalized Verma modules, b-functions, contravariant forms, semisimple Lie algebras.