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Convergence to a viscosity solution for an advection-reaction-diffusion equation arising from a chemotaxis-growth model

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Abstract. We study the limiting behavior as ε tends to zero of the solution of a Cauchy problem for an advection-reaction-diffusion equation; this equation arises in a model for a chemotaxis growth process in biology. We consider the case of an arbitrary time interval and prove the convergence of the solution of this problem to the unique viscosity solution of a limit free boundary problem.

1. Introduction

In this paper, we study the limiting behavior as ε tends to zero of the solution ϕ^ε of an advection-reaction-diffusion equation arising from a chemotaxis-growth model proposed by Mimura and Tsujikawa [10]. We suppose that the density of the chemotactic substance is a known function $v(x,t)$. More precisely, we consider two Cauchy problems. The first one is given by

$$(P_1^\varepsilon)\begin{cases} \phi_t^\varepsilon = \Delta\phi^\varepsilon - \nabla.(\phi^\varepsilon\nabla\chi(v)) + \dfrac{1}{\varepsilon^2}f(\phi^\varepsilon,\varepsilon\alpha) & \text{in } R^N\times(0,T] \\ \phi^\varepsilon(x,0) = \phi_0^\varepsilon(x) & x\in R^N, \end{cases}$$

where $f(s,\tilde{\alpha}) = s(1-s)(s-1/2+\tilde{\alpha})$, and where α is a fixed constant. The functions ϕ^ε and v are respectively the population density and the concentration of chemotactic substance. Here, χ and v are supposed to be smooth functions. The population is subjected to three competitive effects: diffusion, growth induced by the nonlinear term $\phi^\varepsilon(1-\phi^\varepsilon)(\phi^\varepsilon-1/2+\tilde{\alpha})$ and a tendency of migrating towards higher gradients of the chemotactic substance induced by the advection term.

The second problem, that we consider has a slighly different scaling, namely

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