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A note on Yamabe-type equations on the Heisenberg group

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ABSTRACT. We obtain some existence and nonexistence results for the Yamabe-type equation (1.1) on the Heisenberg group, improving recent theorems by Lu and Wei [LW].

1. Introduction

After the fundamental works by Jerison and Lee on the Yamabe problem for CR manifolds, the last few years have shown a growing interest on semilinear equations on the Heisenberg group (see [GL], [Bi], [C], [BC], [BRS], [LW], [BCC1-2], [B1-2], [LU1-2], [U1-2-3], [CC], [CU], [BP]). The Yamabe problem on a CR manifold leads to the following semilinear equation

$$\Delta_{\mathbf{H}^n} u + a(\xi)u + K(\xi)u^p = 0, \qquad \xi = (z,t) \in \mathbf{H}^n, \tag{1.1}$$

where $\Delta_{\mathbf{H}^n}$ denotes the Kohn Laplacian on the Heisenberg group $\mathbf{H}^n$ and $p = 1 + 2/n$ (see [JL1-2-3]). For this reason, for any $p \geq 1$, (1.1) is called by several authors a Yamabe-type equation. Throughout this paper a, K will be supposed to be locally Hölder continuous on $\mathbf{H}^n$.

In this note, by using some recent results in [U2], we obtain some existence and nonexistence results for (1.1) which slightly improve analogous theorems by Lu and Wei in [LW] and by Brandolini, Rigoli and Setti in [BRS]. We give a contribution in the same direction as in the papers [N], [Na], [LN], [CN], in which the classical Laplace-Yamabe equation is studied.

In order to state our results we introduce some notation. In §2, which contains the proofs of our theorems, we give the other definitions and notation we need. Let us denote by $Q = 2n + 2$ the homogeneous dimension of $\mathbf{H}^n$, by d the intrinsic distance and by X the vector field $(z, 2t)$ generating the group of dilations on $\mathbf{H}^n$ (see [GL]). A function $K : \mathbf{H}^n \to \mathbf{R}$ will be said to belong to $L^{]q[}(\mathbf{H}^n)$ if there exist $q_1 < q$ and $q_2 > q$ such that $K \in L^{q_1}(\mathbf{H}^n) \cap L^{q_2}(\mathbf{H}^n)$. Our first result is the following nonexistence theorem that improves Theorem 4.1 in [LW].

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