HIROSHIMA MATH. J.
30 (2000), 345–362

On the action of β_1 in the stable homotopy of spheres at the prime 3

Katsumi SHIMOMURA

(Received May 12, 1999)
(Revised September 20, 1999)

ABSTRACT. Let S^0 denote the sphere spectrum localized away from 3. The element β_1 is the generator of the homotopy group $\pi_{10}(S^0)$. Toda showed that $\beta_1^5 \neq 0$ and $\beta_1^6 = 0$. In this paper, we generalize his result and show that $\beta_1^4\beta_{9t+1} \neq 0$ and $\beta_1^5\beta_{9t+1} = 0$ for $\beta_{9t+1} \in \pi_{144t+10}(S^0)$ with $t \geq 0$. In particular, $\beta_1^4\beta_{10} \neq 0$ and $\beta_1^5\beta_{10} = 0$, where the existence of β_{10} was shown by Oka. This is proved by determining subgroups of $\pi_*(L_2S^0)$. Here L_2 denotes the Bousfield localization functor with respect to $v_2^{-1}BP$.

1. Introduction

Let p be a prime number and S^0 the sphere spectrum localized away from p. Let $E_r^*(X)$ denote the E_r-term of the Adams-Novikov spectral sequence converging to $\pi_*(X)$ for a spectrum X localized away from p. Miller, Ravenel and Wilson [1] introduced β-elements $\beta_{s/j,i+1}$ in $E_2^2(S^0)$ for $(s,j,i+1) \in \boldsymbol{B}^+$, where

$$\boldsymbol{B}^+ = \{(s,j,i+1) \in \boldsymbol{Z}^3 \mid s = mp^n, n \geq 0, p \nmid m \geq 1, j \geq 1, i \geq 0, \text{ subject to}$$
$$\text{i) } j \leq p^n \text{ if } m = 1, \text{ ii) } p^i | j \leq a_{n-i}, \text{ and iii) } a_{n-i-1} < j \text{ if } p^{i+1} | j\}$$

for integers a_k defined by $a_0 = 1$ and $a_k = p^k + p^{k-1} - 1$. Here we use the abbreviation $\beta_{s/j,1} = \beta_{s/j}$ and $\beta_{s/1,1} = \beta_s$.

Let $V(1)$ denote the Toda-Smith spectrum, which is a cofiber of the Adams map $\alpha : \Sigma^{2p-2}V(0) \to V(0)$, where $V(0)$ is the mod p Moore spectrum. Since there exists a map $\beta : \Sigma^{2p^2-2}V(1) \to V(1)$ which induces v_2 on BP-homology at a prime $p > 3$ by [9], we have homotopy elements $\beta_t \in \pi_{2t(p^2-1)-2p}(S^0)$ with $t > 0$. On the other hand, there is no such self map at the prime 3. However there are homotopy elements β_i for $i = 1,2,3,5,6,10$ in this case due to Toda and Oka (*cf.* [2]). Besides, assuming the existence of the self map $B : \Sigma^{144}V(1) \to V(1)$ that induces v_2^9 on BP-homology, we see

---

2000 *Mathematics Subject Classification.* 55Q45, 55Q51.

Key words and phrases. Homotopy groups of spheres, L_2-localization, β-elements, Adams-Novikov spectral sequence.