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Proper Lusternik-Schnirelmann π_1-categories

Antonio REGIDOR-GARCÍA

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ABSTRACT. We define new proper homotopy invariants, the proper Lusternik-Schnirelmann π_1-categories $\mathrm{p}\tilde{\pi}_1$-cat and $\mathrm{p}\tilde{\pi}_1^\infty$-cat. Then, we prove that, if $\mathrm{p}\tilde{\pi}_1$-cat (resp. $\mathrm{p}\tilde{\pi}_1^\infty$-cat) of a locally path-connected, Hausdorff, locally compact, and paracompact space is equal to or less than n, then there is a proper map to a locally finite polyhedron of dimension $n+1$ that induces an isomorphism of fundamental pro-groups $\mathrm{p}\tilde{\pi}_1$ (resp. $\mathrm{p}\tilde{\pi}_1^\infty$).

1. Introduction

The L-S category was defined in 1934 in [12] by L. Lusternik and L. Schnirelmann in the course of their studies on calculus of variations, because it gives a lower bound of the number of critical points of a smooth real function on a closed manifold. The L-S category $\mathrm{cat}\, X$ of a space X is the least number of open subsets contractible in X needed to cover X minus one. It is a homotopy invariant, and was early studied by Borsuk [2] and Fox [7]. Also, there is an algebraic counterpart of the L-S category cat_{π_1} defined by using fundamental groups, due to Fox [7]. The L-S π_1-category $\mathrm{cat}_{\pi_1} X$ of X is the least number of open subsets π_1-contractible in X needed to cover X minus one, where a subset of X is π_1-contractible in X if every loop in the subset is contractible to a point in X. It has been studied for example in [6], [8] and [10].

Homotopy invariants, as cat and cat_{π_1}, do not suffice to study open manifolds, and proper homotopy invariants are needed to investigate the behaviour of these spaces at infinity. Ayala, Domínguez, Márquez, and Quintero [1] have defined a proper version of the L-S category. They have introduced two proper invariants, p-cat and p-cat$^\infty$, using subsets that are properly contractible to the image of the half-line $\mathbf{R}_+$.

In this paper we introduce two new proper homotopy invariants, $\mathrm{p}\tilde{\pi}_1$-cat and $\mathrm{p}\tilde{\pi}_1^\infty$-cat, corresponding to cat_{π_1}. Concretely, $\mathrm{p}\tilde{\pi}_1$-cat coincides with cat_{π_1} for compact spaces. In §2 we define two pro-groups, the fundamental pro-

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