Hiroshima Math. J.
31 (2001), 435–438

Periods of cut-and-project tiling spaces obtained from root lattices

Kazushi Komatsu

(Received January 15, 2001)
(Revised May 2, 2001)

Abstract. Let $\mathcal{T}(E)$ be the tiling space given by the cut-and-project method in $\mathbf{R}^d = E \oplus E^{\perp}$ with a root lattice L. We will consider the dimension n of the linear space of the periods of $\mathcal{T}(E)$. We present a theorem which determines n from the root lattice L in $\mathbf{R}^d$.

1. Introduction

In 1981 de Bruijn [2], [3] introduced the cut-and-project method to construct tilings such as Penrose tilings with icosahedral symmetry. The cut-and-project method was extended to the higher dimensional hypercubic lattices [5] and to more general lattices [6]. To construct tilings and tiling spaces by the cut-and-project method, the hypercubic lattices are most frequently used. However, we are interested in tilings and tiling spaces obtained from root lattices because it is impossible for some tilings to be obtained from hypercubic lattices but possible from root lattices (cf. [1]).

First, we recall the definitions of tilings and tiling spaces by the cut-and-project method (cf. [5], [6], [7], [9]). Let L be a lattice in $\mathbf{R}^d$ with a basis $\{b_i \mid i = 1, 2, \dots, d\}$. Let E be a p-dimensional subspace of $\mathbf{R}^d$, and $E^{\perp}$ its orthogonal complement with respect to the standard inner product. Let $\pi : \mathbf{R}^d \to E$ be the orthogonal projection onto E, and $\pi^{\perp} : \mathbf{R}^d \to E^{\perp}$ the orthogonal projection onto $E^{\perp}$. Let A be a Voronoi cell of L. For any $x \in \mathbf{R}^d$ we put $W_x = \pi^{\perp}(x) + \pi^{\perp}(A) = \{\pi^{\perp}(x) + u \mid u \in \pi^{\perp}(A)\}$, which is called a window for the projection. We define $\Lambda(x)$ by $\Lambda(x) = \pi((W_x \times E) \cap L)$. Let $\mathcal{V}(x)$ denote the Voronoi tiling induced by $\Lambda(x)$, which consists of the Voronoi cells of $\Lambda(x)$. For a vertex v in $\mathcal{V}(x)$ we define $S(v)$ by $S(v) = \bigcup\{P \in \mathcal{V}(x) \mid v \in P\}$. The tiling $T(x)$ given by the cut-and-project method is defined as the collection of tiles $\mathrm{Conv}(S(v) \cap \Lambda(x))$, where $\mathrm{Conv}(B)$ denotes the convex hull of a set B. Note that $\Lambda(x)$ is the set of the vertices of $T(x)$.

2000 *Mathematics Subject Classification*. 37B50, 52C23.
Key words and phrases. period, cut-and-project method, tiling space, root lattice.