Hiroshima Math. J.
32 (2002), 1–6

The Hausdorff dimension of deformed self-similar sets

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(Received October 12, 1999)
(Revised June 12, 2001)

Abstract. We define deformed self-similar sets which are generated by a sequence of similar contraction mappings $\{\phi_\sigma : \sigma \in S^*\}$ on $\mathbf{R^d}$, ϕ_σ having its contraction ratio r_σ, and calculate thier Hausdorff dimension.

1. Introduction

Hutchinson [4] proved that there exists a unique compact set $F \subset \mathbf{R}^d$ such that $F = \bigcup_{i=1}^n \phi_i(F)$ for any given finite set $\{\phi_i\}_{i=1}^n$ of similarities in $\mathbf{R}^d$ with ratio r_i, $1 \le i \le n$. He also showed that $\dim_H F = \dim_B F = \dim_p F = s$ and $\sum_{i=1}^n r_i^s = 1$ if $\{\phi_i\}_{i=1}^n$ satisfies the open set condition, that is, there exists a bounded non-empty open set O such that $\bigcup_{i=1}^n \phi_i(O) \subset O$ and $\phi_i(O) \cap \phi_j(O) = \emptyset$ if $i \ne j$.

Recently, S. Ikeda [5] defined the loosely self-similar set F which is generated by a sequence of mappings $\{\phi_{i_1 i_2 \dots i_k}\}$ ($i_j \in \{1, 2, \dots, n\}$), $\phi_{i_1 i_2 \dots i_k}$ having its contraction ratio r_{i_k}, and showed that $\dim_H F = s$ and $\sum_{i=1}^n r_i^s = 1$ if $\{\phi_{i_1 i_2 \dots i_k}\}$ satisfies the disjoint condition.

In this paper, we will generalize loosely self-similar sets [5] and perturbed Cantor sets [1]. The construction is as follows.

Fix $m \ge 2$, write $S_k = \{1, 2, \dots, m\}^k$ and $S^* = \bigcup_{k=1}^\infty S_k$. Consider a sequence of similar contraction mappings $\{\phi_\sigma : \sigma \in S^*\}$ on $\mathbf{R^d}$. Suppose that each ϕ_σ has a contraction ratio r_σ, that is, $|\phi_\sigma(x) - \phi_\sigma(y)| = r_\sigma |x - y|$ for any $x, y \in \mathbf{R^d}$, where $|\cdot|$ is the Euclidean norm. We further assume there exists $0 < \alpha, \beta < 1$ such that $\alpha < r_\sigma < \beta$ for any $\sigma \in S^*$ and there exists a bounded open set $V \subset \mathbf{R^d}$ such that

(1) $\phi_\sigma(V) \subset V$ for any $\sigma \in S^*$

(2) $\phi_{i_1 i_2 \dots i_{k-1} i_k}(V) \cap \phi_{i_1 i_2 \dots i_{k-1} i_k'}(V) = \emptyset$, $i_k \ne i_k'$.

It is obvious that there exists a non-empty compact set $X \subset V$ such that the properties (1) and (2) are satisfied when V is replaced by X.

2000 *Mathematics Subject Classification*. 28A78, 28A80.

Key words and phrases. Deformed self-similar set, Hausdorff measure, Hausdorff dimension.

The first author was supported by KOSEF and the second author was supported by Com[2] MaC-KOSEF. The third author was supported by TGRC and Korean Research Foundation 1998-015-D00019.