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Parametrizations of infinite biconvex sets in affine root systems

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ABSTRACT. We investigate relationships between the set $\mathfrak{B}^\infty$ of all infinite "biconvex" sets in the positive root system Δ_+ of an arbitrary untwisted affine Lie algebra $\mathfrak{g}$ and the set $\mathscr{W}^\infty$ of all infinite "reduced word" of the Weyl group of $\mathfrak{g}$. The study is applied to the classification of "convex orders" on Δ_+ ([5]), which is indispensable to construct "convex bases" of Poincaré-Birkhoff-Witt type of the strictly upper triangular subalgebra U_q^+ of the quantized universal enveloping algebra $U_q(\mathfrak{g})$. We construct a set $\mathscr{P}$ by using data of the underlying finite-dimensional simple Lie algebra, and bijective mappings $\nabla : \mathscr{P} \to \mathfrak{B}^\infty$ and $\chi : \mathscr{P} \to W^\infty$ such that $\nabla = \Phi^\infty \circ \chi$, where W^∞ is a quotient set of $\mathscr{W}^\infty$ and $\Phi^\infty : W^\infty \to \mathfrak{B}^\infty$ is a natural injective mapping.

1. Introduction

Let Δ be the root system of a Kac-Moody Lie algebra $\mathfrak{g}$, Δ_+ (resp. Δ_-) the set of all positive (resp. negative) roots relative to the root basis $\Pi = \{\alpha_i \mid i \in \mathbf{I}\}$, and $W = \langle s_i \mid i \in \mathbf{I} \rangle$ the Weyl group of $\mathfrak{g}$, where s_i is the reflection associated with α_i. Then (W, S) is a Coxeter system with $S = \{s_i \mid i \in \mathbf{I}\}$ ([6]). We call an infinite sequence $\boldsymbol{s} = (\boldsymbol{s}(p))_{p \in \mathbf{N}} \in S^{\mathbf{N}}$ an *infinite reduced word* of (W, S) if the length of the element $[\boldsymbol{s}|_p] := \boldsymbol{s}(1) \cdots \boldsymbol{s}(p) \in W$ is p for each $p \in \mathbf{N}$, and call a subset $B \subset \Delta_+$ a *biconvex set* if it satisfies the following conditions:

C(i) $\beta, \gamma \in B,\ \beta + \gamma \in \Delta_+ \Rightarrow \beta + \gamma \in B$;

C(ii) $\beta, \gamma \in \Delta_+ \backslash B,\ \beta + \gamma \in \Delta_+ \Rightarrow \beta + \gamma \in \Delta_+ \backslash B$.

If, in addition, B is a subset of the set Δ_+^{re} of all positive real roots, then B is called a *real biconvex set*. The purpose of this article is to investigate in detail relationships between infinite reduced words and infinite real biconvex sets in the case where $\mathfrak{g}$ is an arbitrary untwisted affine Lie algebra.

Before explaining the detail of our work, we will explain the background of the theory of infinite reduced words and infinite real biconvex sets. The motive of this study is related to the construction of *convex bases* of the strictly upper triangular subalgebra U_q^+ of the quantized universal enveloping algebra $U_q(\mathfrak{g})$. Convex bases are Poincaré-Birkhoff-Witt type bases with a convex

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