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Hardy's theorem for the Jacobi transform

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Abstract. Let $\alpha\beta = \frac{1}{4}$ for positive constants α, β. Hardy's theorem states that the function $f(x) = e^{-\alpha x^2}$ is the only function (modulo constants) satisfying the decay conditions $f(x) = O(e^{-\alpha x^2})$ and $\hat{f}(x) = O(e^{-\beta x^2})$, where $\hat{f}$ denotes the Fourier transform of f. We generalise this theorem and its L^p analogues to the Jacobi transform. We then consider the Fourier transform on the real hyperbolic spaces $SO_o(m,n)/SO_o(m-1,n)$, $m,n \in \mathbf{N}$, and show, as an application of our results for the Jacobi transform, that Hardy's theorem only can be generalised to the Riemannian $(m = 1)$ case. It can, in particular, not be generalised to $SL(2,\mathbf{R}) \simeq SU(1,1) \simeq SO_o(2,2)/SO_o(1,2)$.

1. Introduction

Let f be a measurable function on $\mathbf{R}$ and let $\hat{f}$ be its Fourier transform. Assume that $|f(t)| \leq Ce^{-\alpha|t|^2}$ and $|\hat{f}(\lambda)| \leq Ce^{-\beta|\lambda|^2}$, where C, α, β are positive constants. Hardy's theorem, [11], states that if:

(1) $\alpha\beta > \frac{1}{4}$, then $f = 0$.

(2) $\alpha\beta = \frac{1}{4}$, then $f(t) = const.\ e^{-\alpha t^2}$.

(3) $\alpha\beta < \frac{1}{4}$, then there are infinitely many linearly independent solutions.

We note that (2) implies (1) and (3). The central part of Hardy's theorem, the $\alpha\beta = \frac{1}{4}$ case, can be reformulated in terms of the Heat kernel: $h_t(x) := (4\pi t)^{-1/2} e^{-x^2/4t}$, $t > 0$. We note that $\hat{h}_t(x) = e^{-tx^2}$, and thus the only functions satisfying (2) are constant multiples of h_β, with $\beta = 1/4\alpha$. The $\alpha\beta > \frac{1}{4}$ case is also known as Hardy's uncertainty principle: f and $\hat{f}$ cannot both be *very* rapidly decreasing. A Generalisation of Hardy's theorem with L^p growth conditions was furthermore given by Cowling and Price in [6].

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