On Systems of Simultaneous Functional Equations.

By

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The object of this paper is to search for a system of functions that are linearly transformed when their variables are linearly transformed.

(I)

Problem: Under the assumptions that

(i) The equation (1.1) below is satisfied independently of x's and a's,

(ii) $w^{\varkappa\lambda}(x)$, $K^{\varkappa\lambda}_{\mu\nu}(a)$ have first partial derivatives,

to solve the following functional equation

$$w^{\varkappa\lambda}(X) = \sum_{\mu,\nu} K^{\varkappa\lambda}_{\mu\nu}(a) \cdot w^{\mu\nu}(x)\,, \qquad (\varkappa, \lambda = 1, 2, \ldots, n) \tag{1.1}$$

where

$$X^i = \sum_r a^i_r x^r\,, \qquad (i = 1, 2, \ldots, n)$$

and $w^{\varkappa\lambda}(x)$, $K^{\varkappa\lambda}_{\mu\nu}(a)$ represent respectively functions of x's and a's.

Solution: Differentiating (1.1) with a^i_i, we have

$$\frac{\partial w^{\varkappa\lambda}(X)}{\partial X^i} \cdot \frac{\partial X^i}{\partial a^i_i} = \sum_{\mu,\nu} \frac{\partial K^{\varkappa\lambda}_{\mu\nu}(a)}{\partial a^i_i} \cdot w^{\mu\nu}(x)\,. \tag{1.2}$$

Put $a^i_j = \delta^i_j =$ Kronecker's delta, then we have a system of differential equations with respect to x^i,

$$x^i \frac{\partial w^{\varkappa\lambda}(x)}{\partial x^i} = \sum_{\mu,\nu} L^{\varkappa\lambda}_{\mu\nu} \cdot w^{\mu\nu}(x)\,, \tag{1.3}$$

where

$$L^{\varkappa\lambda}_{\mu\nu} = \left[\frac{\partial K^{\varkappa\lambda}_{\mu\nu}}{\partial a^i_i}\right]_{a=\delta}.$$

For the time being, i is considered as being fixed. (1.3) reduces,