Differential Set Functions.

By

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Let $f(x)$ and $g(x)$ be continuous functions in (a, b), moreover $g(x)$ be monotone increasing. If $\lim\sum\frac{|\Delta f|^2}{\Delta g}$ exists, we denoted it by $\int_a^b\frac{|df|^2}{dg}$. For $\int_a^b\frac{|df|^2}{dg}$ to exist, it is necessary and sufficient that $f(x)$ shall be the indefinite Lebesgue-Stieltjes integral of a function $f'_g(x)$ which is integrable with respect to $g(x)$ in (a, b) with its absolute square. And we have

$$(1)\qquad \int_a^b\frac{|df|^2}{dg}=\int_a^b|f'_g(x)|^2dg(x)\,.^{(1)}$$

Put $\beta(E)=g(d)-g(c)$, when E is an open interval (c, d), and $\beta(E)=0$, when E is a point. In like wise we define $\xi(E)$ for $f(x)$. Then $\beta(E)$ and $\xi(E)$ are differential set functions in (a, b). But in the above definition of the Hellinger integral, there are considered only finite decompositions of (a, b), but not arbitrary ones. That $\xi(E)$ is completely additive is at first sight not so obvious, but follows from the inequality $|\Delta f|^2\leqq\Delta g\Delta h$, where $h(x)=\int_a^x\frac{|df|^2}{dg}$. From this we see that $\xi(E)$ is extended to the completely additive set function. These circumstances lead us to specialize the decomposition system. We define the decomposition systems M and M^* in the abstract theory of differential set functions which correspond in the above example to finite decompositions and finite or infinite decompositions of (a, b) respectively. We introduce the concept of complete additivity in $M(M^*)$ in the obvious manner. It is natural, then, to ask whether any differential set function which is completely additive in M may be extended to the ordinary set function.

(1) H. Hahn, Monatshefte für Math. und Physik, **23** (1912), 172.
J. Radon, Sitzungsberichte Wien, **122,** 2 a (1913), 120.