Mathematical Foundations of Quantum Mechanics.

By

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Although quantum mechanics has achieved great success in the explanation of physical phenomena, the mathematics used in quantum mechanics is not rigorous. Unrigorous treatments occur in the case of continuous spectrum. In a Hilbert space, let A be a self-adjoint operator. When we define the eigenvalue and the eigenelement by the relation

$$A\mathfrak{f}=\lambda\mathfrak{f}, \tag{1}$$

we encounter no obstacle in the case of discrete eigenvalues. But if we use the same definition in the case of continuous eigenvalues we encounter a serious obstacle. Let $\mathfrak{f}_\lambda$ be the eigenelement which corresponds to the continuous eigenvalue λ; then the orthogonality of the system of eigenelements is expressed as follows:

$$(\mathfrak{f}_\lambda, \mathfrak{f}_{\lambda'})=\delta(\lambda-\lambda'),$$

where δ is the Dirac improper δ function, defined by

$$\int_{-\infty}^{\infty}\delta(x)\,dx=1$$

$$\delta(x)=0 \qquad \text{for} \quad x\neq 0.$$

Thus we must deal with an improper function. Furthermore, the eigenelements $\mathfrak{f}_\lambda$ do not exist in the Hilbert space.(1)

For example, let the Hilbert space be the space of point functions $f(q)$ (i.e. ordinary wave functions). In the case of the operator which corresponds to the coordinate,

(1) Cf. P. A. M. Dirac [2], 78–79. The numbers in square brackets refer to the bibliography at the end of the paper,