On the Location of the Roots of Linear Combinations of some Polynomials.

By

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(Received Jan. 8, 1937.)

I.

Let α be any complex number and ξ be any root of the equation of order n

$$A(\xi) \equiv f(\xi)+k_1(\xi-\alpha)f'(\xi)+\ \ldots\ldots\ +k_n(\xi-\alpha)^n f^{(n)}(\xi) = 0\,,$$

where $f(z)$ is a given polynomial of order n, then the following equations

$$g(x) \equiv f(\xi)+f'(\xi)x+\ \ldots\ldots\ +\frac{f^{(n)}(\xi)}{n\,!}x^n = 0$$

and

$$h(x) \equiv x^n+nk_1(\alpha-\xi)x^{n-1}+n(n-1)k_2(\alpha-\xi)^2x^{n-2}+\ \ldots\ldots\ +n\,!\ k_n(\alpha-\xi)^n = 0$$

are apolar with each other. Hence by Grace's theorem,(1) $h(x) = 0$ has at least one root in the circle which comprises all the roots of $g(x) = 0$. If we put $x = z-\xi$ in $g(x) = 0$, we have $g(z-\xi) \equiv f(z)$. Thus if the circle C contains all the roots of $f(z) = 0$, then $h(z-\xi) = 0$ has at least one root within C. Rewriting $h(z-\xi) = 0$ in the form

$$K(y) \equiv y^n+nk_1y^{n-1}+n(n-1)k_2y^{n-2}+\ \ldots\ldots\ +n\,!\ k_n = 0 \qquad \left(y = \frac{z-\xi}{\alpha-\xi}\right),$$

we obtain

Theorem I. Let z be a suitable point in the circle C containing

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(1) After the idea used by Prof. T. Takagi in his "Note on the algebraic equations." Proc. Phys.-Math. soc. of Japan, (3), **3** (1921), 176, we start from the theorem due to J. H. Grace.

(2) In this paper, we use the word "circle" to mean the "Kreisbereich" in G. Pólya und G. Szegö, *Aufgaben und Lehrsatze aus der Analysis.* II. 55.